CHAPTER

2 Consumer Choice

2.A Introduction

The most fundamental decision unit of microeconomic theory is the *consumer*. In this chapter, we begin our study of consumer demand in the context of a market economy. By a *market economy*, we mean a setting in which the goods and services that the consumer may acquire are available for purchase at known prices (or, equivalently, are available for trade for other goods at known rates of exchange).

We begin, in Sections 2.B to 2.D, by describing the basic elements of the consumer's decision problem. In Section 2.B, we introduce the concept of *commodities*, the objects of choice for the consumer. Then, in Sections 2.C and 2.D, we consider the physical and economic constraints that limit the consumer's choices. The former are captured in the *consumption set*, which we discuss in Section 2.C; the latter are incorporated in Section 2.D into the consumer's *Walrasian budget set*.

The consumer's decision subject to these constraints is captured in the consumer's *Walrasian demand function*. In terms of the choice-based approach to individual decision making introduced in Section 1.C, the Walrasian demand function is the consumer's choice rule. We study this function and some of its basic properties in Section 2.E. Among them are what we call *comparative statics* properties: the ways in which consumer demand changes when economic constraints vary.

Finally, in Section 2.F, we consider the implications for the consumer's demand function of the *weak axiom of revealed preference*. The central conclusion we reach is that in the consumer demand setting, the weak axiom is essentially equivalent to the *compensated law of demand*, the postulate that prices and demanded quantities move in opposite directions for price changes that leave real wealth unchanged.

2.B Commodities

The decision problem faced by the consumer in a market economy is to choose consumption levels of the various goods and services that are available for purchase in the market. We call these goods and services *commodities*. For simplicity, we assume that the number of commodities is finite and equal to L (indexed by $\ell = 1, \dots, L$).

As a general matter, a *commodity vector* (or commodity *bundle*) is a list of amounts of the different commodities,

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix},$$

and can be viewed as a point in $\mathbb{R}^L$, the commodity *space*.[1]

We can use commodity vectors to represent an individual's consumption levels. The ℓth entry of the commodity vector stands for the amount of commodity ℓ consumed. We then refer to the vector as a *consumption vector* or *consumption bundle*.

Note that time (or, for that matter, location) can be built into the definition of a commodity. Rigorously, bread today and tomorrow should be viewed as distinct commodities. In a similar vein, when we deal with decisions under uncertainty in Chapter 6, viewing bread in different "states of nature" as different commodities can be most helpful.

Although commodities consumed at different times should be viewed rigorously as distinct commodities, in practice, economic models often involve some "time aggregation." Thus, one commodity might be "bread consumed in the month of February," even though, in principle, bread consumed at each instant in February should be distinguished. A primary reason for such time aggregation is that the economic data to which the model is being applied are aggregated in this way. The hope of the modeler is that the commodities being aggregated are sufficiently similar that little of economic interest is being lost.

We should also note that in some contexts it becomes convenient, and even necessary, to expand the set of commodities to include goods and services that may potentially be available for purchase but are not actually so and even some that may be available by means other than market exchange (say, the experience of "family togetherness"). For nearly all of what follows here, however, the narrow construction introduced in this section suffices.

2.C The Consumption Set

Consumption choices are typically limited by a number of physical constraints. The simplest example is when it may be impossible for the individual to consume a negative amount of a commodity such as bread or water.

Formally, the *consumption set* is a subset of the commodity space $\mathbb{R}^L$, denoted by $X \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume given the physical constraints imposed by his environment.

Consider the following four examples for the case in which $L = 2$:

(i) Figure 2.C.1 represents possible consumption levels of bread and leisure in a day. Both levels must be nonnegative and, in addition, the consumption of more than 24 hours of leisure in a day is impossible.

(ii) Figure 2.C.2 represents a situation in which the first good is perfectly divisible but the second is available only in nonnegative integer amounts.

(iii) Figure 2.C.3 captures the fact that it is impossible to eat bread at the same

1. Negative entries in commodity vectors will often represent debits or net outflows of goods. For example, in Chapter 5, the inputs of a firm are measured as negative numbers.

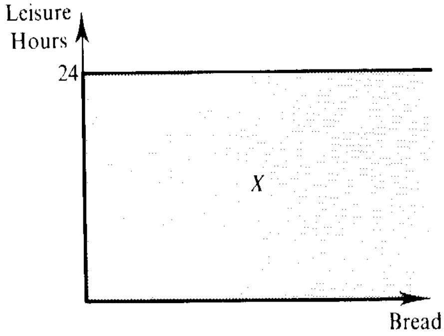


Figure 2.C.1 (left)
A consumption set.

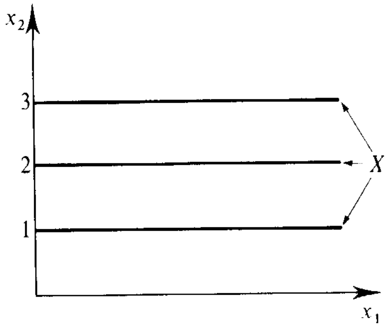


Figure 2.C.2 (right)
A consumption set where good 2 must be consumed in integer amounts.

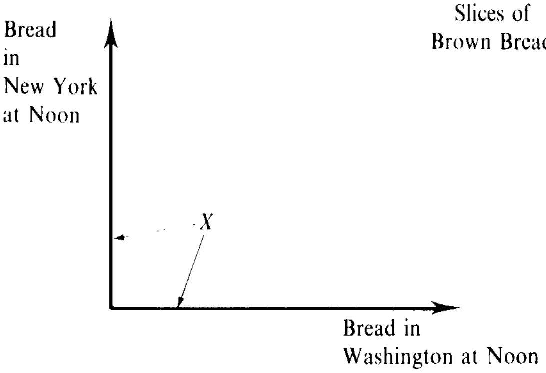


Figure 2.C.3 (left)
A consumption set where only one good can be consumed.

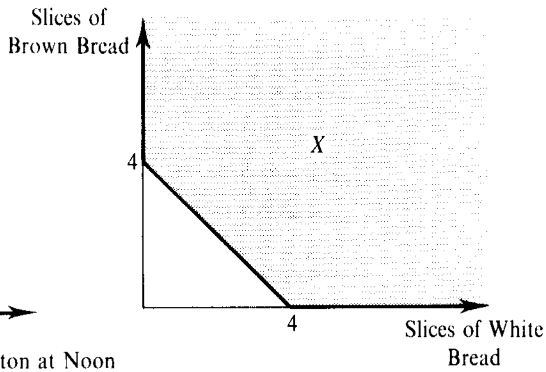


Figure 2.C.4 (right)
A consumption set reflecting survival needs.

instant in Washington and in New York. [This example is borrowed from Malinvaud (1978).]

(iv) Figure 2.C.4 represents a situation where the consumer requires a minimum of four slices of bread a day to survive and there are two types of bread, brown and white.

In the four examples, the constraints are physical in a very literal sense. But the constraints that we incorporate into the consumption set can also be institutional in nature. For example, a law requiring that no one work more than 16 hours a day would change the consumption set in Figure 2.C.1 to that in Figure 2.C.5.

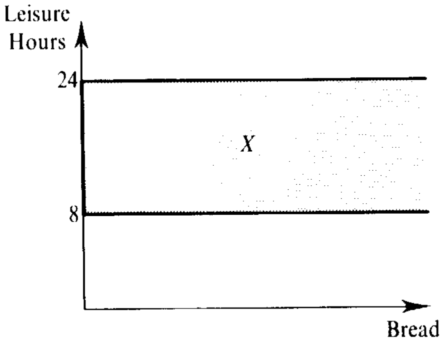
To keep things as straightforward as possible, we pursue our discussion adopting the simplest sort of consumption set:

$$X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_\ell \geq 0 \text{ for } \ell = 1, \dots, L\},$$

the set of all nonnegative bundles of commodities. It is represented in Figure 2.C.6. Whenever we consider any consumption set X other than $\mathbb{R}^L_+$, we shall be explicit about it.

One special feature of the set $\mathbb{R}^L_+$ is that it is *convex*. That is, if two consumption bundles x and x' are both elements of $\mathbb{R}^L_+$, then the bundle $x'' = \alpha x + (1 - \alpha)x'$ is also an element of $\mathbb{R}^L_+$ for any $\alpha \in [0, 1]$ (see Section M.G. of the Mathematical Appendix for the definition and properties of convex sets).[2] The consumption sets

2. Recall that $x'' = \alpha x + (1 - \alpha)x'$ is a vector whose ℓth entry is $x''_\ell = \alpha x_\ell + (1 - \alpha)x'_\ell$.



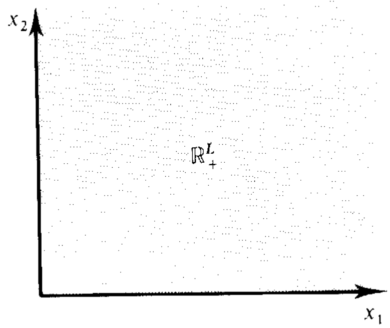


Figure 2.C.5 (left) A consumption set reflecting a legal limit on the number of hours worked.

Figure 2.C.6 (right) The consumption set $\mathbb{R}^L_+$.

in Figures 2.C.1, 2.C.4, 2.C.5, and 2.C.6 are convex sets; those in Figures 2.C.2 and 2.C.3 are not.

Much of the theory to be developed applies for general convex consumption sets as well as for $\mathbb{R}^L_+$. Some of the results, but not all, survive without the assumption of convexity.[3]

2.D Competitive Budgets

In addition to the physical constraints embodied in the consumption set, the consumer faces an important economic constraint: his consumption choice is limited to those commodity bundles that he can afford.

To formalize this constraint, we introduce two assumptions. First, we suppose that the L commodities are all traded in the market at dollar prices that are publicly quoted (this is the *principle of completeness*, or *universality, of markets*). Formally, these prices are represented by the *price vector*

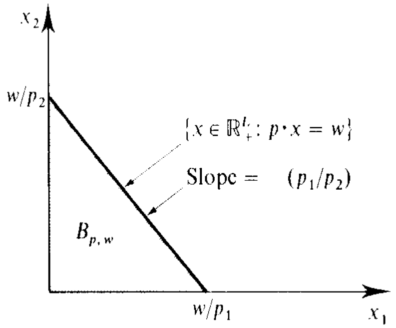
$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L,$$

which gives the dollar cost for a unit of each of the L commodities. Observe that there is nothing that logically requires prices to be positive. A negative price simply means that a "buyer" is actually paid to consume the commodity (which is not illogical for commodities that are "bads," such as pollution). Nevertheless, for simplicity, here we always assume $p \gg 0$; that is, $p_\ell > 0$ for every ℓ.

Second, we assume that these prices are beyond the influence of the consumer. This is the so-called *price-taking assumption*. Loosely speaking, this assumption is likely to be valid when the consumer's demand for any commodity represents only a small fraction of the total demand for that good.

The affordability of a consumption bundle depends on two things: the market prices $p = (p_1, \ldots, p_L)$ and the consumer's wealth level (in dollars) w. The consumption

3. Note that commodity aggregation can help convexify the consumption set. In the example leading to Figure 2.C.3, the consumption set could reasonably be taken to be convex if the axes were instead measuring bread consumption over a period of a month.



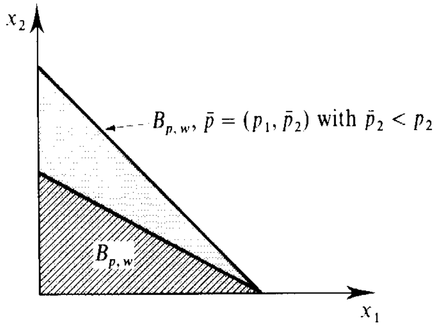


Figure 2.D.1 (left)
A Walrasian budget set.

Figure 2.D.2 (right)
The effect of a price change on the Walrasian budget set.

bundle $x \in \mathbb{R}^L_+$ is affordable if its total cost does not exceed the consumer's wealth level w, that is, if[4]

$$p \cdot x = p_1 x_1 + \cdots + p_L x_L \leq w.$$

This economic-affordability constraint, when combined with the requirement that x lie in the consumption set $\mathbb{R}^L_+$, implies that the set of feasible consumption bundles consists of the elements of the set $\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$. This set is known as the *Walrasian , or competitive budget set* (after Léon Walras).

Definition 2.D.1: The *Walrasian, or competitive budget set* $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

The *consumer's problem*, given prices p and wealth w, can thus be stated as follows: *Choose a consumption bundle x from $B_{p,w}$.*

A Walrasian budget set $B_{p,w}$ is depicted in Figure 2.D.1 for the case of $L = 2$. To focus on the case in which the consumer has a nondegenerate choice problem, we always assume $w > 0$ (otherwise the consumer can afford only $x = 0$).

The set $\{x \in \mathbb{R}^L : p \cdot x = w\}$ is called the *budget hyperplane* (for the case $L = 2$, we call it the *budget line*). It determines the upper boundary of the budget set. As Figure 2.D.1 indicates, the slope of the budget line when $L = 2$, $-(p_1/p_2)$, captures the rate of exchange between the two commodities. If the price of commodity 2 decreases (with p_1 and w held fixed), say to $\bar{p}_2 < p_2$, the budget set grows larger because more consumption bundles are affordable, and the budget line becomes steeper. This change is shown in Figure 2.D.2.

Another way to see how the budget hyperplane reflects the relative terms of exchange between commodities comes from examining its geometric relation to the price vector p. The price vector p, drawn starting from any point $\bar{x}$ on the budget hyperplane, must be orthogonal (perpendicular) to any vector starting at $\bar{x}$ and lying

4. Often, this constraint is described in the literature as requiring that the cost of planned purchases not exceed the consumer's *income*. In either case, the idea is that the cost of purchases not exceed the consumer's available resources. We use the wealth terminology to emphasize that the consumer's actual problem may be intertemporal, with the commodities involving purchases over time, and the resource constraint being one of lifetime income (i.e., wealth) (see Exercise 2.D.1).

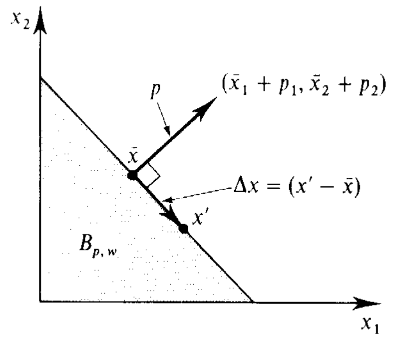


Figure 2.D.3
The geometric relationship between p and the budget hyperplane.

on the budget hyperplane, This is so because for any x' that itself lies on the budget hyperplane, we have $p \cdot x' = p \cdot \bar{x} = w$. Hence, $p \cdot \Delta x = 0$ for $\Delta x = (x' - \bar{x})$. Figure 2.D.3 depicts this geometric relationship for the case $L = 2$.[5]

The Walrasian budget set $B_{p,w}$ is a *convex* set: That is, if bundles x and x' are both elements of $B_{p,w}$, then the bundle $x'' = \alpha x + (1 - \alpha)x'$ is also. To see this, note first that because both x and x' are nonnegative, $x'' \in \mathbb{R}^L_+$. Second, since $p \cdot x \leq w$ and $p \cdot x' \leq w$, we have $p \cdot x'' = \alpha(p \cdot x) + (1 - \alpha)(p \cdot x') \leq w$. Thus, $x'' \in B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$.

The convexity of $B_{p,w}$ plays a significant role in the development that follows. Note that the convexity of $B_{p,w}$ depends on the convexity of the consumption set $\mathbb{R}^L_+$. With a more general consumption set X, $B_{p,w}$ will be convex as long as X is. (See Exercise 2.D.3.)

Although Walrasian budget sets are of central theoretical interest, they are by no means the only type of budget set that a consumer might face in any actual situation. For example, a more realistic description of the market trade-off between a consumption good and leisure, involving taxes, subsidies, and several wage rates, is illustrated in Figure 2.D.4. In the figure, the price of the consumption good is 1, and the consumer earns wage rate s per hour for the first 8 hours of work and $s' > s$ for additional ("overtime") hours. He also faces a tax rate t

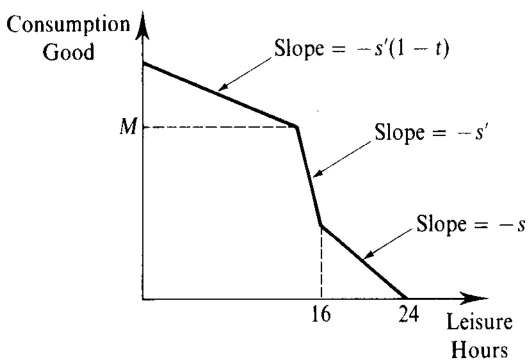


Figure 2.D.4
A more realistic description of the consumer's budget set.

5. To draw the vector p starting from $\bar{x}$, we draw a vector from point $(\bar{x}_1, \bar{x}_2)$ to point $(\bar{x}_1 + p_1, \bar{x}_2 + p_2)$. Thus, when we draw the price vector in this diagram, we use the "units" on the axes to represent units of prices rather than goods.

per dollar on labor income earned above amount M. Note that the budget set in Figure 2.D.4 is not convex (you are asked to show this in Exercise 2.D.4). More complicated examples can readily be constructed and arise commonly in applied work. See Deaton and Muellbauer (1980) and Burtless and Hausmann (1975) for more illustrations of this sort.

2.E Demand Functions and Comparative Statics

The consumer's *Walrasian* (or *market*, or *ordinary*) *demand correspondence* $x(p, w)$ assigns a set of chosen consumption bundles for each price–wealth pair (p, w). In principle, this correspondence can be multivalued; that is, there may be more than one possible consumption vector assigned for a given price–wealth pair (p, w). When this is so, any $x \in x(p, w)$ might be chosen by the consumer when he faces price–wealth pair (p, w). When $x(p, w)$ is single-valued, we refer to it as a *demand function.*

Throughout this chapter, we maintain two assumptions regarding the Walrasian demand correspondence $x(p, w)$: That it is *homogeneous of degree zero* and that it satisfies *Walras' law.*

Definition 2.E.1: The Walrasian demand correspondence $x(p, w)$ is *homogeneous of degree zero* if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

Homogeneity of degree zero says that if both prices and wealth change in the same proportion, then the individual's consumption choice does not change. To understand this property, note that a change in prices and wealth from (p, w) to $(\alpha p, \alpha w)$ leads to no change in the consumer's set of feasible consumption bundles; that is, $B_{p,w} = B_{\alpha p, \alpha w}$. Homogeneity of degree zero says that the individual's choice depends only on the set of feasible points.

Definition 2.E.2: The Walrasian demand correspondence $x(p, w)$ satisfies Walras' law if for every $p \gg 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p, w)$.

Walras' law says that the consumer fully expends his wealth. Intuitively, this is a reasonable assumption to make as long as there is some good that is clearly desirable. Walras' law should be understood broadly: the consumer's budget may be an intertemporal one allowing for savings today to be used for purchases tomorrow. What Walras' law says is that the consumer fully expends his resources *over his lifetime.*

Exercise 2.E.1: Suppose $L = 3$, and consider the demand function $x(p, w)$ defined by

$$x_1(p, w) = \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1},$$

$$x_2(p, w) = \frac{p_3}{p_1 + p_2 + p_3} \frac{w}{p_2},$$

$$x_3(p, w) = \frac{\beta p_1}{p_1 + p_2 + p_3} \frac{w}{p_3}.$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when $\beta = 1$? What about when $\beta \in (0,1)$?

In Chapter 3, where the consumer's demand $x(p, w)$ is derived from the maximization of preferences, these two properties (homogeneity of degree zero and satisfaction of Walras' law) hold under very general circumstances. In the rest of this chapter, however, we shall simply take them as assumptions about $x(p, w)$ and explore their consequences.

One convenient implication of $x(p, w)$ being homogeneous of degree zero can be noted immediately: Although $x(p, w)$ formally has $L + 1$ arguments, we can, with no loss of generality, fix (*normalize*) the level of one of the $L + 1$ independent variables at an arbitrary level. One common normalization is $p_\ell = 1$ for some ℓ. Another is $w = 1$.[6] Hence, the effective number of arguments in $x(p, w)$ is L.

For the remainder of this section, we assume that $x(p, w)$ is always single-valued. In this case, we can write the function $x(p, w)$ in terms of commodity-specific demand functions:

$$x(p, w) = \begin{bmatrix} x_1(p, w) \\ x_2(p, w) \\ \vdots \\ x_L(p, w) \end{bmatrix}.$$

When convenient, we also assume $x(p, w)$ to be continuous and differentiable.

The approach we take here and in Section 2.F can be viewed as an application of the choice-based framework developed in Chapter 1. The family of Walrasian budget sets is $\mathscr{B}^w = \{B_{p,w}: p \gg 0, w > 0\}$. Moreover, by homogeneity of degree zero, $x(p, w)$ depends only on the budget set the consumer faces. Hence $(\mathscr{B}^w, x(\cdot))$ is a choice structure, as defined in Section 1.C. Note that the choice structure $(\mathscr{B}^w, x(\cdot))$ does not include all possible subsets of X (e.g., it does not include all two- and three-element subsets of X). This fact will be significant for the relationship between the choice-based and preference-based approaches to consumer demand.

Comparative Statics

We are often interested in analyzing how the consumer's choice varies with changes in his wealth and in prices. The examination of a change in outcome in response to a change in underlying economic parameters is known as *comparative statics* analysis.

Wealth effects

For fixed prices $\bar{p}$, the function of wealth $x(\bar{p}, w)$ is called the consumer's *Engel function*. Its image in $\mathbb{R}^L_+$, $E_{\bar{p}} = \{x(\bar{p}, w): w > 0\}$, is known as the *wealth expansion path*. Figure 2.E.1 depicts such an expansion path.

At any (p, w), the derivative $\partial x_\ell(p, w)/\partial w$ is known as the *wealth effect* for the ℓth good.[7]

6. We use normalizations extensively in Part IV.

7. It is also known as the *income effect* in the literature. Similarly, the wealth expansion path is sometimes referred to as an *income expansion path*.

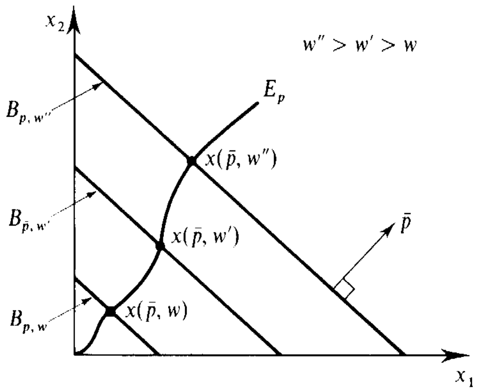


Figure 2.E.1
The wealth expansion path at prices $\bar{p}$.

A commodity ℓ is *normal* at (p, w) if $\partial x_\ell(p, w)/\partial w \geq 0$; that is, demand is nondecreasing in wealth. If commodity ℓ's wealth effect is instead negative, then it is called *inferior* at (p, w). If every commodity is normal at all (p, w), then we say that *demand is normal.*

The assumption of normal demand makes sense if commodities are large aggregates (e.g., food, shelter). But if they are very disaggregated (e.g., particular kinds of shoes), then because of substitution to higher-quality goods as wealth increases, goods that become inferior at some level of wealth may be the rule rather than the exception.

In matrix notation, the wealth effects are represented as follows:

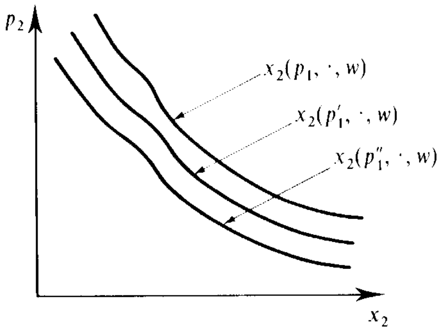
$$D_w x(p, w) = \begin{bmatrix} \dfrac{\partial x_1(p, w)}{\partial w} \\ \dfrac{\partial x_2(p, w)}{\partial w} \\ \vdots \\ \dfrac{\partial x_L(p, w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L.$$

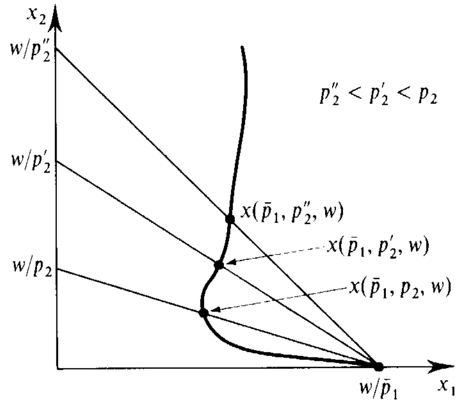
Price effects

We can also ask how consumption levels of the various commodities change as prices vary.

Consider first the case where $L = 2$, and suppose we keep wealth and price p_1 fixed. Figure 2.E.2 represents the demand function for good 2 as a function of its own price p_2 for various levels of the price of good 1, with wealth held constant at amount w. Note that, as is customary in economics, the price variable, which here is the independent variable, is measured on the vertical axis, and the quantity demanded, the dependent variable, is measured on the horizontal axis. Another useful representation of the consumers' demand at different prices is the locus of points demanded in $\mathbb{R}^2_+$ as we range over all possible values of p_2. This is known as an *offer curve*. An example is presented in Figure 2.E.3.

More generally, the derivative $\partial x_\ell(p, w)/\partial p_k$ is known as the *price effect of* p_k, the price of good k, *on the demand for good* ℓ. Although it may be natural to think that a fall in a good's price will lead the consumer to purchase more of it (as in





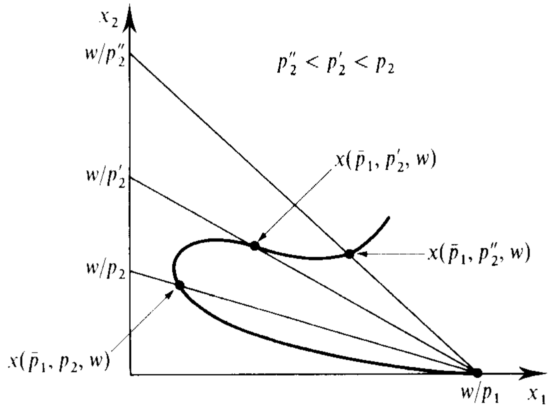


Figure 2.E.2 (top left)
The demand for good 2 as a function of its price (for various levels of p_1).

Figure 2.E.3 (top right)
An offer curve.

Figure 2.E.4 (bottom)
An offer curve where good 2 is inferior at $(\bar{p}, p'_2, w)$.

Figure 2.E.3), the reverse situation is not an economic impossibility. Good ℓ is said to be a *Giffen good* at (p, w) if $\partial x_\ell(p, w)/\partial p_\ell > 0$. For the offer curve depicted in Figure 2.E.4, good 2 is a Giffen good at $(\bar{p}_1, p'_2, w)$.

Low-quality goods may well be Giffen goods for consumers with low wealth levels. For example, imagine that a poor consumer initially is fulfilling much of his dietary requirements with potatoes because they are a low-cost way to avoid hunger. If the price of potatoes falls, he can then afford to buy other, more desirable foods that also keep him from being hungry. His consumption of potatoes may well fall as a result. Note that the mechanism that leads to potatoes being a Giffen good in this story involves a wealth consideration: When the price of potatoes falls, the consumer is effectively wealthier (he can afford to purchase more generally), and so he buys fewer potatoes. We will be investigating this interplay between price and wealth effects more extensively in the rest of this chapter and in Chapter 3.

The price effects are conveniently represented in matrix form as follows:

$$D_p x(p, w) = \begin{bmatrix} \dfrac{\partial x_1(p, w)}{\partial p_1} & \cdots & \dfrac{\partial x_1(p, w)}{\partial p_L} \\ & \ddots & \\ \dfrac{\partial x_L(p, w)}{\partial p_1} & \cdots & \dfrac{\partial x_L(p, w)}{\partial p_L} \end{bmatrix}.$$

Implications of homogeneity and Walras' law for price and wealth effects

Homogeneity and Walras' law imply certain restrictions on the comparative statics effects of consumer demand with respect to prices and wealth.

Consider, first, the implications of homogeneity of degree zero. We know that $x(\alpha p, \alpha w) - x(p, w) = 0$ for all $\alpha > 0$. Differentiating this expression with respect to α, and evaluating the derivative at $\alpha = 1$, we get the results shown in Proposition 2.E.1 (the result is also a special case of Euler's formula; see Section M.B of the Mathematical Appendix for details).

Proposition 2.E.1: If the Walrasian demand function $x(p, w)$ is homogeneous of degree zero, then for all p and w:

$$\sum_{k=1}^{L} \frac{\partial x_\ell(p, w)}{\partial p_k} p_k + \frac{\partial x_\ell(p, w)}{\partial w} w = 0 \text{ for } \ell = 1, \ldots, L. \tag{2.E.1}$$

In matrix notation, this is expressed as

$$D_p x(p, w) p + D_w x(p, w) w = 0. \tag{2.E.2}$$

Thus, homogeneity of degree zero implies that the price and wealth derivatives of demand for any good ℓ, when weighted by these prices and wealth, sum to zero. Intuitively, this weighting arises because when we increase all prices and wealth proportionately, each of these variables changes in proportion to its initial level.

We can also restate equation (2.E.1) in terms of the *elasticities* of demand with respect to prices and wealth. These are defined, respectively, by

$$\varepsilon_{\ell k}(p, w) = \frac{\partial x_\ell(p, w)}{\partial p_k} \frac{p_k}{x_\ell(p, w)}$$

and

$$\varepsilon_{\ell w}(p, w) = \frac{\partial x_\ell(p, w)}{\partial w} \frac{w}{x_\ell(p, w)}.$$

These elasticities give the *percentage* change in demand for good ℓ per (marginal) percentage change in the price of good k or wealth; note that the expression for $\varepsilon_{\ell w}(\cdot, \cdot)$ can be read as $(\Delta x/x)/(\Delta w/w)$. Elasticities arise very frequently in applied work. Unlike the derivatives of demand, elasticities are independent of the units chosen for measuring commodities and therefore provide a unit-free way of capturing demand responsiveness.

Using elasticities, condition (2.E.1) takes the following form:

$$\sum_{k=1}^{L} \varepsilon_{\ell k}(p, w) + \varepsilon_{\ell w}(p, w) = 0 \quad \text{for } \ell = 1, \ldots, L. \tag{2.E.3}$$

This formulation very directly expresses the comparative statics implication of homogeneity of degree zero: An equal percentage change in all prices and wealth leads to no change in demand.

Walras' law, on the other hand, has two implications for the price and wealth effects of demand. By Walras' law, we know that $p \cdot x(p, w) = w$ for all p and w. Differentiating this expression with respect to prices yields the first result, presented in Proposition 2.E.2.

Proposition 2.E.2: If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p, w)}{\partial p_k} + x_k(p, w) = 0 \quad \text{for } k = 1, \ldots, L, \tag{2.E.4}$$

or, written in matrix notion,[8]

$$p \cdot D_p x(p, w) + x(p, w)^{\mathrm{T}} = 0^{\mathrm{T}}. \tag{2.E.5}$$

Similarly, differentiating $p \cdot x(p, w) = w$ with respect to w, we get the second result, shown in Proposition 2.E.3.

Proposition 2.E.3: If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p, w)}{\partial w} = 1, \tag{2.E.6}$$

or, written in matrix notation,

$$p \cdot D_w x(p, w) = 1. \tag{2.E.7}$$

The conditions derived in Propositions 2.E.2 and 2.E.3 are sometimes called the properties of *Cournot* and *Engel aggregation*, respectively. They are simply the differential versions of two facts: That total expenditure cannot change in response to a change in prices and that total expenditure must change by an amount equal to any wealth change.

Exercise 2.E.2: Show that equations (2.E.4) and (2.E.6) lead to the following two elasticity formulas:

$$\sum_{\ell=1}^{L} b_\ell(p, w)\varepsilon_{\ell k}(p, w) + b_k(p, w) = 0,$$

and

$$\sum_{\ell=1}^{L} b_\ell(p, w)\varepsilon_{\ell w}(p, w) = 1,$$

where $b_\ell(p, w) = p_\ell x_\ell(p, w)/w$ is the budget share of the consumer's expenditure on good ℓ given prices p and wealth w.

2.F The Weak Axiom of Revealed Preference and the Law of Demand

In this section, we study the implications of the weak axiom of revealed preference for consumer demand. Throughout the analysis, we continue to assume that $x(p, w)$ is single-valued, homogeneous of degree zero, and satisfies Walras' law.[9]

The weak axiom was already introduced in Section 1.C as a consistency axiom for the choice-based approach to decision theory. In this section, we explore its implications for the demand behavior of a consumer. In the preference-based approach to consumer behavior to be studied in Chapter 3, demand necessarily

8. Recall that 0^{T} means a row vector of zeros.
9. For generalizations to the case of multivalued choice, see Exercise 2.F.13.

satisfies the weak axiom. Thus, the results presented in Chapter 3, when compared with those in this section, will tell us how much more structure is imposed on consumer demand by the preference-based approach beyond what is implied by the weak axiom alone.[10]

In the context of Walrasian demand functions, the weak axiom takes the form stated in the Definition 2.F.1.

Definition 2.F.1: The Walrasian demand function $x(p, w)$ satisfies the *weak axiom of revealed preference* (the WA) if the following property holds for any two price-wealth situations (p, w) and (p', w'):

$$\text{If } p \cdot x(p', w') \leq w \quad \text{and} \quad x(p', w') \neq x(p, w), \quad \text{then} \quad p' \cdot x(p, w) > w'.$$

If you have already studied Chapter 1, you will recognize that this definition is precisely the specialization of the general statement of the weak axiom presented in Section 1.C to the context in which budget sets are Walrasian and $x(p, w)$ specifies a unique choice (see Exercise 2.F.1).

In the consumer demand setting, the idea behind the weak axiom can be put as follows: If $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$, then we know that when facing prices p and wealth w, the consumer chose consumption bundle $x(p, w)$ even though bundle $x(p', w')$ was also affordable. We can interpret this choice as "revealing" a preference for $x(p, w)$ over $x(p', w')$. Now, we might reasonably expect the consumer to display some consistency in his demand behavior. In particular, given his revealed preference, we expect that he would choose $x(p, w)$ over $x(p', w')$ whenever they are both affordable. If so, bundle $x(p, w)$ must not be affordable at the price-wealth combination (p', w') at which the consumer chooses bundle $x(p', w')$. That is, as required by the weak axiom, we must have $p' \cdot x(p, w) > w'$.

The restriction on demand behavior imposed by the weak axiom when $L = 2$ is illustrated in Figure 2.F.1. Each diagram shows two budget sets $B_{p',w'}$ and $B_{p'',w''}$ and their corresponding choice $x(p', w')$ and $x(p'', w'')$. The weak axiom tells us that we cannot have both $p' \cdot x(p'', w'') \leq w'$ and $p'' \cdot x(p', w') \leq w''$. Panels (a) to (c) depict permissible situations, whereas demand in panels (d) and (e) violates the weak axiom.

Implications of the Weak Axiom

The weak axiom has significant implications for the effects of price changes on demand. We need to concentrate, however, on a special kind of price change.

As the discussion of Giffen goods in Section 2.E suggested, price changes affect the consumer in two ways. First, they alter the relative cost of different commodities. But, second, they also change the consumer's real wealth: An increase in the price of a commodity impoverishes the consumers of that commodity. To study the implications of the weak axiom, we need to isolate the first effect.

One way to accomplish this is to imagine a situation in which a change in prices is accompanied by a change in the consumer's wealth that makes his initial consumption bundle just affordable at the new prices. That is, if the consumer is originally facing prices p and wealth w and chooses consumption bundle $x(p, w)$, then

10. Or, stated more properly, beyond what is implied by the weak axiom in conjunction with homogeneity of degree zero and Walras' law.

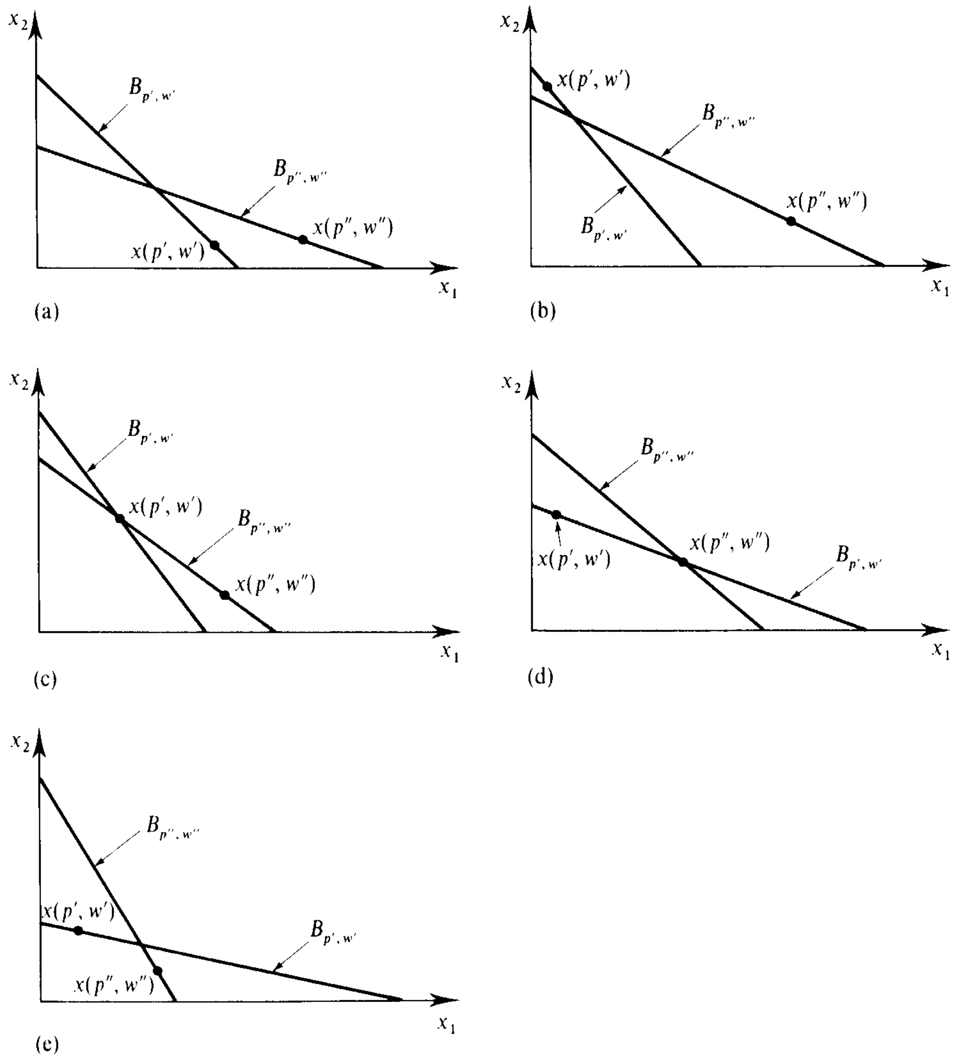


Figure 2.F.1
Demand in panels (a) to (c) satisfies the weak axiom; demand in panels (d) and (e) does not.

when prices change to p', we imagine that the consumer's wealth is adjusted to $w' = p' \cdot x(p, w)$. Thus, the wealth adjustment is $\Delta w = \Delta p \cdot x(p, w)$, where $\Delta p = (p' - p)$. This kind of wealth adjustment is known as *Slutsky wealth compensation.* Figure 2.F.2 shows the change in the budget set when a reduction in the price of good 1 from p_1 to p'_1 is accompanied by Slutsky wealth compensation. Geometrically, the restriction is that the budget hyperplane corresponding to (p', w') goes through the vector $x(p, w)$.

We refer to price changes that are accompanied by such compensating wealth changes as *(Slutsky) compensated price changes.*

In Proposition 2.F.1, we show that the weak axiom can be equivalently stated in terms of the demand response to compensated price changes.

Proposition 2.F.1: Suppose that the Walrasian demand function $x(p, w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom if and only if the following property holds:

For any compensated price change from an initial situation (p, w) to a new price wealth pair $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0, \tag{2.F.1}$$

with strict inequality whenever $x(p, w) \neq x(p', w')$.

Proof: (i) *The weak axiom implies inequality (2.F.1), with strict inequality if* $x(p, w) \neq x(p', w')$. The result is immediate if $x(p', w') = x(p, w)$, since then $(p' - p) \cdot [x(p', w') - x(p, w)] = 0$. So suppose that $x(p', w') \neq x(p, w)$. The left-hand side of inequality (2.F.1) can be written as

$$(p' - p) \cdot [x(p', w') - x(p, w)] = p' \cdot [x(p', w') - x(p, w)] - p \cdot [x(p', w') - x(p, w)]. \tag{2.F.2}$$

Consider the first term of (2.F.2). Because the change from p to p' is a compensated price change, we know that $p' \cdot x(p, w) = w'$. In addition, Walras' law tells us that $w' = p' \cdot x(p', w')$. Hence

$$p' \cdot [x(p', w') - x(p, w)] = 0. \tag{2.F.3}$$

Now consider the second term of (2.F.2). Because $p' \cdot x(p, w) = w'$, $x(p, w)$ is affordable under price wealth situation (p', w'). The weak axiom therefore implies that $x(p', w')$ must *not* be affordable under price–wealth situation (p, w). Thus, we must have $p \cdot x(p', w') > w$. Since $p \cdot x(p, w) = w$ by Walras' law, this implies that

$$p \cdot [x(p', w') - x(p, w)] > 0 \tag{2.F.4}$$

Together, (2.F.2), (2.F.3) and (2.F.4) yield the result.

(ii) *The weak axiom is implied by (2.F.1) holding for all compensated price changes, with strict inequality if* $x(p, w) \neq x(p', w')$. The argument for this direction of the proof uses the following fact: The weak axiom holds if and only if it holds for all *compensated* price changes. That is, the weak axiom holds if, for any two price–wealth pairs (p, w) and (p', w'), we have $p' \cdot x(p, w) > w'$ whenever $p \cdot x(p', w') = w$ and $x(p', w') \neq x(p, w)$.

To prove the fact stated in the preceding paragraph, we argue that if the weak axiom is violated, then there must be a compensated price change for which it is violated. To see this, suppose that we have a violation of the weak axiom, that is, two price–wealth pairs (p', w') and (p'', w'') such that $x(p', w') \neq x(p'', w'')$, $p' \cdot x(p'', w'') \leq w'$, and $p'' \cdot x(p', w') \leq w''$. If one of these two weak inequalities holds with equality, then this is actually a compensated price change and we are done. So assume that, as shown in Figure 2.F.3, we have $p' \cdot x(p'', w'') < w'$ and $p'' \cdot x(p', w') < w''$.

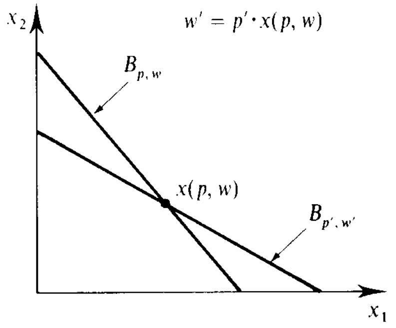


Figure 2.F.2
A compensated price change from (p, w) to (p', w').

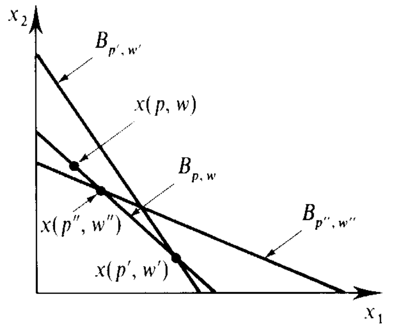


Figure 2.F.3
The weak axiom holds if and only if it holds for all compensated price changes.

Now choose the value of $\alpha \in (0,1)$ for which

$$(\alpha p' + (1-\alpha)p'')\cdot x(p', w') = (\alpha p' + (1-\alpha)p'')\cdot x(p'', w''),$$

and denote $p = \alpha p' + (1-\alpha)p''$ and $w = (\alpha p' + (1-\alpha)p'')\cdot x(p', w')$. This construction is illustrated in Figure 2.F.3. We then have

$$\begin{aligned}\alpha w' + (1-\alpha)w'' &> \alpha p'\cdot x(p', w') + (1-\alpha)p''\cdot x(p', w') \\ &= w \\ &= p\cdot x(p, w) \\ &= \alpha p'\cdot x(p, w) + (1-\alpha)p''\cdot x(p, w).\end{aligned}$$

Therefore, either $p'\cdot x(p, w) < w'$ or $p''\cdot x(p, w) < w''$. Suppose that the first possibility holds (the argument is identical if it is the second that holds). Then we have $x(p, w) \neq x(p', w')$, $p\cdot x(p', w') = w$, and $p'\cdot x(p, w) < w'$, which constitutes a violation of the weak axiom for the compensated price change from (p', w') to (p, w).

Once we know that in order to test for the weak axiom it suffices to consider only compensated price changes, the remaining reasoning is straightforward. If the weak axiom does not hold, there exists a compensated price change from some (p', w') to some (p, w) such that $x(p, w) \neq x(p', w')$, $p\cdot x(p', w') = w$, and $p'\cdot x(p, w) \leq w'$. But since $x(\cdot,\cdot)$ satisfies Walras' law, these two inequalities imply

$$p\cdot[x(p', w') - x(p, w)] = 0 \qquad \text{and} \qquad p'\cdot[x(p', w') - x(p, w)] \geq 0.$$

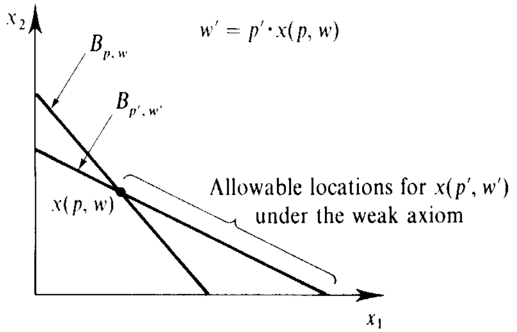
Hence, we would have

$$(p' - p)\cdot[x(p', w') - x(p, w)] \geq 0 \qquad \text{and} \qquad x(p, w) \neq x(p', w'),$$

which is a contradiction to (2.F.1) holding for all compensated price changes [and with strict inequality when $x(p, w) \neq x(p', w')$]. ■

The inequality (2.F.1) can be written in shorthand as $\Delta p\cdot\Delta x \leq 0$, where $\Delta p = (p' - p)$ and $\Delta x = [x(p', w') - x(p, w)]$. It can be interpreted as a form of the *law of demand*: *Demand and price move in opposite directions.* Proposition 2.F.1 tells us that the law of demand holds for *compensated* price changes. We therefore call it the *compensated law of demand.*

The simplest case involves the effect on demand for some good ℓ of a compensated change in its own price p_ℓ. When only this price changes, we have $\Delta p = (0, \ldots, 0, \Delta p_\ell, 0, \ldots, 0)$. Since $\Delta p\cdot\Delta x = \Delta p_\ell\, \Delta x_\ell$, Proposition 2.F.1 tells us that if $\Delta p_\ell > 0$, then we must have $\Delta x_\ell < 0$. The basic argument is illustrated in Figure 2.F.4. Starting at



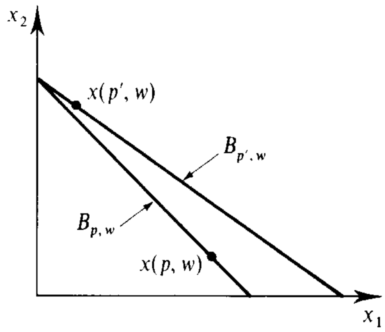


Figure 2.F.4 (left)
Demand must be nonincreasing in own price for a compensated price change.

Figure 2.F.5 (right)
Demand for good 1 can fall when its price decreases for an uncompensated price change.

(p, w), a compensated decrease in the price of good 1 rotates the budget line through $x(p, w)$. The WA allows moves of demand only in the direction that increases the demand of good 1.

Figure 2.F.5 should persuade you that the WA (or, for that matter, the preference maximization assumption discussed in Chapter 3) is not sufficient to yield the law of demand for price changes that are *not* compensated. In the figure, the price change from p to p' is obtained by a decrease in the price of good 1, but the weak axiom imposes no restriction on where we place the new consumption bundle; as drawn, the demand for good 1 falls.

When consumer demand $x(p, w)$ is a differentiable function of prices and wealth, Proposition 2.F.1 has a differential implication that is of great importance. Consider, starting at a given price wealth pair (p, w), a differential change in prices dp. Imagine that we make this a compensated price change by giving the consumer compensation of $dw = x(p, w) \cdot dp$ [this is just the differential analog of $\Delta w = x(p, w) \cdot \Delta p$]. Proposition 2.F.1 tells us that

$$dp \cdot dx \leq 0. \tag{2.F.5}$$

Now, using the chain rule, the differential change in demand induced by this compensated price change can be written as

$$dx = D_p x(p, w)\, dp + D_w x(p, w)\, dw. \tag{2.F.6}$$

Hence

$$dx = D_p x(p, w)\, dp + D_w x(p, w)\, [x(p, w) \cdot dp] \tag{2.F.7}$$

or equivalently

$$dx = [D_p x(p, w) + D_w x(p, w) x(p, w)^{\mathrm{T}}]\, dp. \tag{2.F.8}$$

Finally, substituting (2.F.8) into (2.F.5) we conclude that for any possible differential price change dp, we have

$$dp \cdot [D_p x(p, w) + D_w x(p, w) x(p, w)^{\mathrm{T}}]\, dp \leq 0. \tag{2.F.9}$$

The expression in square brackets in condition (2.F.9) is an $L \times L$ matrix, which we denote by $S(p, w)$. Formally

$$S(p, w) = \begin{bmatrix} s_{11}(p, w) & \cdots & s_{1L}(p, w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p, w) & \cdots & s_{LL}(p, w) \end{bmatrix},$$

where the (ℓ, k)th entry is

$$s_{\ell k}(p, w) = \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w). \qquad (2.F.10)$$

The matrix $S(p, w)$ is known as the *substitution*, or *Slutsky*, matrix, and its elements are known as *substitution effects*.

The "substitution" terminology is apt because the term $s_{\ell k}(p, w)$ measures the differential change in the consumption of commodity ℓ (i.e., the substitution to or from other commodities) due to a differential change in the price of commodity k when wealth is adjusted so that the consumer can still just afford his original consumption bundle (i.e., due solely to a change in relative prices). To see this, note that the change in demand for good ℓ if wealth is left unchanged is $(\partial x_l(p, w)/\partial p_k)\, dp_k$. For the consumer to be able to "just afford" his original consumption bundle, his wealth must vary by the amount $x_k(p, w)\, dp_k$. The effect of this wealth change on the demand for good ℓ is then $(\partial x_\ell(p, w)/\partial w)\, [x_k(p, w)\, dp_k]$. The sum of these two effects is therefore exactly $s_{\ell k}(p, w)\, dp_k$.

We summarize the derivation in equations (2.F.5) to (2.F.10) in Proposition 2.F.2.

Proposition 2.F.2: If a differentiable Walrasian demand function $x(p, w)$ satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any (p, w), the Slutsky matrix $S(p, w)$ satisfies $v \cdot S(p, w) v \leq 0$ for any $v \in \mathbb{R}^L$.

A matrix satisfying the property in Proposition 2.F.2 is called *negative semidefinite* (it is negative *definite* if the inequality is strict for all $v \neq 0$). See Section M.D of the Mathematical Appendix for more on these matrices.

Note that $S(p, w)$ being negative semidefinite implies that $s_{\ell\ell}(p, w) \leq 0$: That is, *the substitution effect of good ℓ with respect to its own price is always nonpositive.*

An interesting implication of $s_{\ell\ell}(p, w) \leq 0$ is that a good can be a Giffen good at (p, w) only if it is inferior. In particular, since

$$s_{\ell\ell}(p, w) = \partial x_\ell(p, w)/\partial p_\ell + [\partial x_\ell(p, w)/\partial w]\, x_\ell(p, w) \leq 0,$$

if $\partial x_\ell(p, w)/\partial p_\ell > 0$, we must have $\partial x_\ell(p, w)/\partial w < 0$.

For later reference, we note that Proposition 2.F.2 does *not* imply, in general, that the matrix $S(p, w)$ is symmetric.[11] For $L = 2$, $S(p, w)$ is necessarily symmetric (you are asked to show this in Exercise 2.F.11). When $L > 2$, however, $S(p, w)$ need not be symmetric under the assumptions made so far (homogeneity of degree zero, Walras' law, and the weak axiom). See Exercises 2.F.10 and 2.F.15 for examples. In Chapter 3 (Section 3.H), we shall see that the symmetry of $S(p, w)$ is intimately connected with the possibility of generating demand from the maximization of rational preferences.

Exploiting further the properties of homogeneity of degree zero and Walras' law, we can say a bit more about the substitution matrix $S(p, w)$.

11. A matter of terminology: It is common in the mathematical literature that "definite" matrices are assumed to be symmetric. Rigorously speaking, if no symmetry is implied, the matrix would be called "quasidefinite." To simplify terminology, we use "definite" without any supposition about symmetry; if a matrix is symmetric, we say so explicitly. (See Exercise 2.F.9.)

Proposition 2.F.3: Suppose that the Walrasian demand function $x(p, w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then $p \cdot S(p, w) = 0$ and $S(p, w)p = 0$ for any (p, w).

Exercise 2.F.7: Prove Proposition 2.F.3. [*Hint*: Use Propositions 2.E.1 to 2.E.3.]

It follows from Proposition 2.F.3 that the matrix $S(p, w)$ is always singular (i.e., it has rank less than L), and so the negative semidefiniteness of $S(p, w)$ established in Proposition 2.F.2 cannot be extended to negative definiteness (e.g., see Exercise 2.F.17).

Proposition 2.F.2 establishes negative semidefiniteness of $S(p, w)$ as a necessary implication of the weak axiom. One might wonder: Is this property sufficient to imply the WA [so that negative semidefiniteness of $S(p, w)$ is actually equivalent to the WA]? That is, if we have a demand function $x(p, w)$ that satisfies Walras' law, homogeneity of degree zero and has a negative semidefinite substitution matrix, must it satisfy the weak axiom? The answer is *almost, but not quite*. Exercise 2.F.16 provides an example of a demand function with a negative semidefinite substitution matrix that violates the WA. The sufficient condition is that $v \cdot S(p, w)v < 0$ whenever $v \neq \alpha p$ for any scalar α; that is, $S(p, w)$ must be negative definite for all vectors other than those that are proportional to p. This result is due to Samuelson [see Samuelson (1947) or Kihlstrom, Mas-Colell, and Sonnenschein (1976) for an advanced treatment]. The gap between the necessary and sufficient conditions is of the same nature as the gap between the necessary and the sufficient second-order conditions for the minimization of a function.

Finally, how would a theory of consumer demand that is based solely on the assumptions of homogeneity of degree zero, Walras' law, and the consistency requirement embodied in the weak action compare with one based on rational preference maximization?

Based on Chapter 1, you might hope that Proposition 1.D.2 implies that the two are equivalent. But we cannot appeal to that proposition here because the family of Walrasian budgets does not include every possible budget; in particular, it does not include all the budgets formed by only two- or three-commodity bundles.

In fact, the two theories are not equivalent. For Walrasian demand functions, the theory derived from the weak axiom is weaker than the theory derived from rational preferences, in the sense of implying fewer restrictions. This is shown formally in Chapter 3, where we demonstrate that if demand is generated from preferences, or is capable of being so generated, then it must have a symmetric Slutsky matrix at all (p, w). But for the moment, Example 2.F.1, due originally to Hicks (1956), may be persuasive enough.

Example 2.F.1: In a three-commodity world, consider the three budget sets determined by the price vectors $p^1 = (2, 1, 2)$, $p^2 = (2, 2, 1)$, $p^3 = (1, 2, 2)$ and wealth $= 8$ (the same for the three budgets). Suppose that the respective (unique) choices are $x^1 = (1, 2, 2)$, $x^2 = (2, 1, 2)$, $x^3 = (2, 2, 1)$. In Exercise 2.F.2, you are asked to verify that any two pairs of choices satisfy the WA but that x^3 is revealed preferred to x^2, x^2 is revealed preferred to x^1, and x^1 is revealed preferred to x^3. This situation is incompatible with the existence of underlying rational preferences (transitivity would be violated).

The reason this example is only *persuasive* and does not quite settle the question is that demand has been defined only for the three given budgets, therefore, we cannot be sure that it satisfies the requirements of the WA for all possible competitive budgets. To clinch the matter we refer to Chapter 3. ■

In summary, there are three primary conclusions to be drawn from Section 2.F:

(i) The consistency requirement embodied in the weak axiom (combined with the homogeneity of degree zero and Walras' law) is equivalent to the compensated law of demand.

(ii) The compensated law of demand, in turn, implies negative semidefiniteness of the substitution matrix $S(p, w)$.

(iii) These assumptions do *not* imply symmetry of $S(p, w)$, except in the case where $L = 2$.

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EXERCISES

2.D.1$^{\text{A}}$ A consumer lives for two periods, denoted 1 and 2, and consumes a single consumption good in each period. His wealth when born is $w > 0$. What is his (lifetime) Walrasian budget set?

2.D.2$^{\text{A}}$ A consumer consumes one consumption good x and hours of leisure h. The price of the consumption good is p, and the consumer can work at a wage rate of $s = 1$. What is the consumer's Walrasian budget set?

2.D.3$^{\text{B}}$ Consider an extension of the Walrasian budget set to an arbitrary consumption set X: $B_{p,w} = \{x \in X : p \cdot x \leq w\}$. Assume $(p, w) \gg 0$.

(a) If X is the set depicted in Figure 2.C.3, would $B_{p,w}$ be convex?

(b) Show that if X is a convex set, then $B_{p,w}$ is as well.

2.D.4$^{\text{A}}$ Show that the budget set in Figure 2.D.4 is not convex.

2.E.1$^{\text{A}}$ In text.

2.E.2$^{\text{B}}$ In text.

2.E.3$^{\text{B}}$ Use Propositions 2.E.1 to 2.E.3 to show that $p \cdot D_p x(p, w)\, p = -w$. Interpret.

2.E.4$^{\text{B}}$ Show that if $x(p, w)$ is homogeneous of degree one with respect to w [i.e., $x(p, \alpha w) = \alpha x(p, w)$

for all $\alpha > 0$] and satisfies Walras' law, then $\varepsilon_{\ell w}(p, w) = 1$ for every ℓ. Interpret. Can you say something about $D_w x(p, w)$ and the form of the Engel functions and curves in this case?

2.E.5[B] Suppose that $x(p, w)$ is a demand function which is homogeneous of degree one with respect to w and satisfies Walras' law and homogeneity of degree zero. Suppose also that all the cross-price effects are zero, that is $\partial x_\ell(p, w)/\partial p_k = 0$ whenever $k \neq \ell$. Show that this implies that for every ℓ, $x_\ell(p, w) = \alpha_\ell w/p_\ell$, where $\alpha_\ell > 0$ is a constant independent of (p, w).

2.E.6[A] Verify that the conclusions of Propositions 2.E.1 to 2.E.3 hold for the demand function given in Exercise 2.E.1 when $\beta = 1$.

2.E.7[A] A consumer in a two-good economy has a demand function $x(p, w)$ that satisfies Walras' law. His demand function for the first good is $x_1(p, w) = \alpha w/p_1$. Derive his demand function for the second good. Is his demand function homogeneous of degree zero?

2.E.8[B] Show that the elasticity of demand for good ℓ with respect to price p_k, $\varepsilon_{\ell k}(p, w)$, can be written as $\varepsilon_{\ell k}(p, w) = d \ln (x_\ell(p, w))/d \ln (p_k)$, where $\ln(\cdot)$ is the natural logarithm function. Derive a similar expression for $\varepsilon_{\ell w}(p, w)$. Conclude that if we estimate the parameters $(\alpha_0, \alpha_1, \alpha_2, \gamma)$ of the equation $\ln (x_\ell(p, w)) = \alpha_0 + \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \gamma \ln w$, these parameter estimates provide us with estimates of the elasticities $\varepsilon_{\ell 1}(p, w)$, $\varepsilon_{\ell 2}(p, w)$, and $\varepsilon_{\ell w}(p, w)$.

2.F.1[B] Show that for Walrasian demand functions, the definition of the weak axiom given in Definition 2.F.1 coincides with that in Definition 1.C.1.

2.F.2[B] Verify the claim of Example 2.F.1.

2.F.3[B] You are given the following partial information about a consumer's purchases. He consumes only two goods.

	Year 1		Year 2	
	Quantity	Price	Quantity	Price
Good 1	100	100	120	100
Good 2	100	100	?	80

Over what range of quantities of good 2 consumed in year 2 would you conclude:

(a) That his behaviour is inconsistent (i.e., in contradiction with the weak axiom)?

(b) That the consumer's consumption bundle in year 1 is revealed preferred to that in year 2?

(c) That the consumer's consumption bundle in year 2 is revealed preferred to that in year 1?

(d) That there is insufficient information to justify (a), (b), and/or (c)?

(e) That good 1 is an inferior good (at some price) for this consumer? Assume that the weak axiom is satisfied.

(f) That good 2 is an inferior good (at some price) for this consumer? Assume that the weak axiom is satisfied.

2.F.4[A] Consider the consumption of a consumer in two different periods, period 0 and period 1. Period t prices, wealth, and consumption are p^t, w_t, and $x^t = x(p^t, w_t)$, respectively. It is often of applied interest to form an index measure of the quantity consumed by a consumer. The *Laspeyres* quantity index computes the change in quantity using period 0 prices as weights: $L_Q = (p^0 \cdot x^1)/(p^0 \cdot x^0)$. The *Paasche* quantity index instead uses period 1 prices as weights: $P_Q = (p^1 \cdot x^1)/(p^1 \cdot x^0)$. Finally, we could use the consumer's expenditure change: $E_Q = (p^1 \cdot x^1)/(p^0 \cdot x^0)$. Show the following:

(a) If $L_Q < 1$, then the consumer has a revealed preference for x^0 over x^1.

(b) If $P_Q > 1$, then the consumer has a revealed preference for x^1 over x^0.

(c) No revealed preference relationship is implied by either $E_Q > 1$ or $E_Q < 1$. Note that at the aggregate level, E_Q corresponds to the percentage change in gross national product.

2.F.5^C Suppose that $x(p, w)$ is a differentiable demand function that satisfies the weak axiom, Walras' law, and homogeneity of degree zero. Show that if $x(\cdot,\cdot)$ is homogeneous of degree one with respect to w [i.e., $x(p, \alpha w) = \alpha x(p, w)$ for all (p, w) and $\alpha > 0$], then the law of demand holds even for *uncompensated* price changes. If this is easier, establish only the infinitesimal version of this conclusion; that is, $dp \cdot D_p x(p, w)\, dp \leq 0$ for any dp.

2.F.6^A Suppose that $x(p, w)$ is homogeneous of degree zero. Show that the weak axiom holds if and only if for some $w > 0$ and all p, p' we have $p' \cdot x(p, w) > w$ whenever $p \cdot x(p', w) \leq w$ and $x(p', w) \neq x(p, w)$.

2.F.7^B In text.

2.F.8^A Let $\hat{s}_{\ell k}(p, w) = [p_k / x_\ell(p, w)] s_{\ell k}(p, w)$ be the substitution terms in elasticity form. Express $\hat{s}_{\ell k}(p, w)$ in terms of $\varepsilon_{\ell k}(p, w)$, $\varepsilon_{\ell w}(p, w)$, and $b_k(p, w)$.

2.F.9^B A symmetric $n \times n$ matrix A is negative definite if and only if $(-1)^k |A_{kk}| > 0$ for all $k \leq n$, where A_{kk} is the submatrix of A obtained by deleting the last $n - k$ rows and columns. For semidefiniteness of the symmetric matrix A, we replace the strict inequalities by weak inequalities and require that the weak inequalities hold for all matrices formed by permuting the rows and columns of A (see Section M.D of the Mathematical Appendix for details).

(a) Show that an arbitrary (possibly nonsymmetric) matrix A is negative definite (or semidefinite) if and only if $A + A^T$ is negative definite (or semidefinite). Show also that the above determinant condition (which can be shown to be necessary) is no longer sufficient in the nonsymmetric case.

(b) Show that for $L = 2$, the necessary and sufficient condition for the substitution matrix $S(p, w)$ of rank 1 to be negative semidefinite is that any diagonal entry (i.e., any own-price substitution effect) be negative.

2.F.10^B Consider the demand function in Exercise 2.E.1 with $\beta = 1$. Assume that $w = 1$.

(a) Compute the substitution matrix. Show that at $p = (1, 1, 1)$, it is negative semidefinite but not symmetric.

(b) Show that this demand function does not satisfy the weak axiom. [*Hint*: Consider the price vector $p = (1, 1, \varepsilon)$ and show that the substitution matrix is not negative semidefinite (for $\varepsilon > 0$ small).]

2.F.11^A Show that for $L = 2$, $S(p, w)$ is always symmetric. [*Hint*: Use Proposition 2.F.3.]

2.F.12^A Show that if the Walrasian demand function $x(p, w)$ is generated by a rational preference relation, than it must satisfy the weak axiom.

2.F.13^C Suppose that $x(p, w)$ may be multivalued.

(a) From the definition of the weak axiom given in Section 1.C, develop the generalization of Definition 2.F.1 for Walrasian demand correspondences.

(b) Show that if $x(p, w)$ satisfies this generalization of the weak axiom and Walras' law, then $x(\cdot)$ satisfies the following property:

(*) For any $x \in x(p, w)$ and $x' \in x(p', w')$, if $p \cdot x' < w$, then $p \cdot x > w$.

(c) Show that the generalized weak axiom and Walras' law implies the following generalized version of the compensated law of demand: Starting from any initial position (p, w) with demand $x \in x(p, w)$, for any compensated price change to new prices p' and wealth level $w' = p' \cdot x$, we have

$$(p' - p) \cdot (x' - x) \leq 0$$

for all $x' \in x(p', w')$, with strict inequality if $x' \in x(p, w)$.

(d) Show that if $x(p, w)$ satisfies Walras' law and the generalized compensated law of demand defined in **(c)**, then $x(p, w)$ satisfies the generalized weak axiom.

2.F.14[A] Show that if $x(p, w)$ is a Walrasian demand function that satisfies the weak axiom, then $x(p, w)$ must be homogeneous of degree zero.

2.F.15[B] Consider a setting with $L = 3$ and a consumer whose consumption set is $\mathbb{R}^3$. The consumer's demand function $x(p, w)$ satisfies homogeneity of degree zero, Walras' law and (fixing $p_3 = 1$) has

$$x_1(p, w) = -p_1 + p_2$$

and

$$x_2(p, w) = -p_2.$$

Show that this demand function satisfies the weak axiom by demonstrating that its substitution matrix satisfies $v \cdot S(p, w)\, v < 0$ for all $v \neq \alpha p$. [*Hint*: Use the matrix results recorded in Section M.D of the Mathematical Appendix.] Observe then that the substitution matrix is not symmetric. (*Note*: The fact that we allow for negative consumption levels here is not essential for finding a demand function that satisfies the weak axiom but whose substitution matrix is not symmetric; with a consumption set allowing only for nonnegative consumption levels, however, we would need to specify a more complicated demand function.)

2.F.16[B] Consider a setting where $L = 3$ and a consumer whose consumption set is $\mathbb{R}^3$. Suppose that his demand function $x(p, w)$ is

$$x_1(p, w) = \frac{p_2}{p_3},$$

$$x_2(p, w) = -\frac{p_1}{p_3},$$

$$x_3(p, w) = \frac{w}{p_3}.$$

(a) Show that $x(p, w)$ is homogeneous of degree zero in (p, w) and satisfies Walras' law.

(b) Show that $x(p, w)$ violates the weak axiom.

(c) Show that $v \cdot S(p, w)\, v = 0$ for all $v \in \mathbb{R}^3$.

2.F.17[B] In an L-commodity world, a consumer's Walrasian demand function is

$$x_k(p, w) = \frac{w}{\left(\sum_{\ell=1}^{L} p_\ell\right)} \quad \text{for } k = 1, \ldots, L.$$

(a) Is this demand function homogeneous of degree zero in (p, w)?

(b) Does it satisfy Walras' law?

(c) Does it satisfy the weak axiom?

(d) Compute the Slutsky substitution matrix for this demand function. Is it negative semidefinite? Negative definite? Symmetric?