3 Classical Demand Theory

3.A Introduction

In this chapter, we study the classical, preference-based approach to consumer demand.

We begin in Section 3.B by introducing the consumer's preference relation and some of its basic properties. We assume throughout that this preference relation is *rational*, offering a complete and transitive ranking of the consumer's possible consumption choices. We also discuss two properties, *monotonicity* (or its weaker version, *local nonsatiation*) and *convexity*, that are used extensively in the analysis that follows.

Section 3.C considers a technical issue: the existence and continuity properties of utility functions that represent the consumer's preferences. We show that not all preference relations are representable by a utility function, and we then formulate an assumption on preferences, known as *continuity*, that is sufficient to guarantee the existence of a (continuous) utility function.

In Section 3.D, we begin our study of the consumer's decision problem by assuming that there are L commodities whose prices she takes as fixed and independent of her actions (the *price-taking assumption*). The consumer's problem is framed as one of *utility maximization* subject to the constraints embodied in the Walrasian budget set. We focus our study on two objects of central interest: the consumer's optimal choice, embodied in the *Walrasian* (or *market* or *ordinary*) *demand correspondence*, and the consumer's optimal utility value, captured by the *indirect utility function*.

Section 3.E introduces the consumer's *expenditure minimization problem*, which bears a close relation to the consumer's goal of utility maximization. In parallel to our study of the demand correspondence and value function of the utility maximization problem, we study the equivalent objects for expenditure minimization. They are known, respectively, as the *Hicksian* (or *compensated*) *demand correspondence* and the *expenditure function*. We also provide an initial formal examination of the relationship between the expenditure minimization and utility maximization problems.

In Section 3.F, we pause for an introduction to the mathematical underpinnings of duality theory. This material offers important insights into the structure of preference-based demand theory. Section 3.F may be skipped without loss of continuity in a first reading of the chapter. Nevertheless, we recommend the study of its material.

Section 3.G continues our analysis of the utility maximization and expenditure minimization problems by establishing some of the most important results of demand theory. These results develop the fundamental connections between the demand and value functions of the two problems.

In Section 3.H, we complete the study of the implications of the preference-based theory of consumer demand by asking how and when we can recover the consumer's underlying preferences from her demand behavior, an issue traditionally known as the *integrability problem*. In addition to their other uses, the results presented in this section tell us that the properties of consumer demand identified in Sections 3.D to 3.G as *necessary* implications of preference-maximizing behavior are also *sufficient* in the sense that any demand behavior satisfying these properties can be rationalized as preference-maximizing behavior.

The results in Sections 3.D to 3.H also allow us to compare the implications of the preference-based approach to consumer demand with the choice-based theory studied in Section 2.F. Although the differences turn out to be slight, the two approaches are not equivalent; the choice-based demand theory founded on the weak axiom of revealed preference imposes fewer restrictions on demand than does the preference-based theory studied in this chapter. The extra condition added by the assumption of rational preferences turns out to be the *symmetry* of the Slutsky matrix. As a result, we conclude that satisfaction of the weak axiom does not ensure the existence of a rationalizing preference relation for consumer demand.

Although our analysis in Sections 3.B to 3.H focuses entirely on the positive (i.e., descriptive) implications of the preference-based approach, one of the most important benefits of the latter is that it provides a framework for normative, or *welfare*, analysis. In Section 3.I, we take a first look at this subject by studying the effects of a price change on the consumer's welfare. In this connection, we discuss the use of the traditional concept of Marshallian surplus as a measure of consumer welfare.

We conclude in Section 3.J by returning to the choice-based approach to consumer demand. We ask whether there is some strengthening of the weak axiom that leads to a choice-based theory of consumer demand equivalent to the preference-based approach. As an answer, we introduce the *strong axiom of revealed preference* and show that it leads to demand behavior that is consistent with the existence of underlying preferences.

Appendix A discusses some technical issues related to the continuity and differentiability of Walrasian demand.

For further reading, see the thorough treatment of classical demand theory offered by Deaton and Muellbauer (1980).

3.B Preference Relations: Basic Properties

In the classical approach to consumer demand, the analysis of consumer behavior begins by specifying the consumer's preferences over the commodity bundles in the consumption set $X \subset \mathbb{R}^L_+$.

The consumer's preferences are captured by a preference relation $\succsim$ (an "at-least-as-good-as" relation) defined on X that we take to be *rational* in the sense introduced in Section 1.B; that is, $\succsim$ is *complete* and *transitive*. For convenience, we repeat the formal statement of this assumption from Definition 1.B.1.[1]

Definition 3.B.1: The preference relation $\succsim$ on X is *rational* if it possesses the following two properties:

(i) *Completeness*. For all $x, y \in X$, we have $x \succsim y$ or $y \succsim x$ (or both).
(ii) *Transitivity*. For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

In the discussion that follows, we also use two other types of assumptions about preferences: *desirability* assumptions and *convexity* assumptions.

(i) *Desirability assumptions*. It is often reasonable to assume that larger amounts of commodities are preferred to smaller ones. This feature of preferences is captured in the assumption of monotonicity. For Definition 3.B.2, we assume that the consumption of larger amounts of goods is always feasible in principle; that is, if $x \in X$ and $y \geq x$, then $y \in X$.

Definition 3.B.2: The preference relation $\succsim$ on X is *monotone* if $x \in X$ and $y \gg x$ implies $y \succ x$. It is *strongly monotone* if $y \geq x$ and $y \neq x$ imply that $y \succ x$.

The assumption that preferences are monotone is satisfied as long as commodities are "goods" rather than "bads". Even if some commodity is a bad, however, we may still be able to view preferences as monotone because it is often possible to redefine a consumption activity in a way that satisfies the assumption. For example, if one commodity is garbage, we can instead define the individual's consumption over the "absence of garbage".[2]

Note that if $\succsim$ is monotone, we may have indifference with respect to an increase in the amount of some but not all commodities. In contrast, strong montonicity says that if y is larger than x for *some* commodity and is no less for any other, then y is strictly preferred to x.

For much of the theory, however, a weaker desirability assumption than monotonicity, known as *local nonsatiation*, actually suffices.

Definition 3.B.3: The preference relation $\succsim$ on X is *locally nonsatiated* if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.[3]

The test for locally nonsatiated preferences is depicted in Figure 3.B.1 for the case in which $X = \mathbb{R}^L_+$. It says that for any consumption bundle $x \in \mathbb{R}^L_+$ and any arbitrarily

1. See Section 1.B for a thorough discussion of these properties.

2. It is also sometimes convenient to view preferences as defined over the level of goods available for consumption (the stocks of goods on hand), rather than over the consumption levels themselves. In this case, if the consumer can freely dispose of any unwanted commodities, her preferences over the level of commodities on hand are monotone as long as some good is always desirable.

3. $\|x - y\|$ is the Euclidean distance between points x and y; that is, $\|x - y\| = [\sum_{\ell=1}^{L} (x_\ell - y_\ell)^2]^{1/2}$.

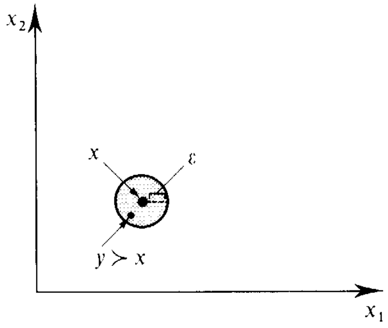


Figure 3.B.1
The test for local nonsatiation.

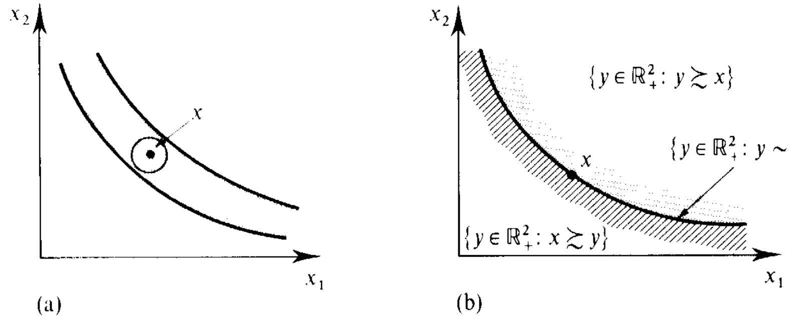


Figure 3.B.2
(a) A thick indifference set violates local nonsatiation.
(b) Preferences compatible with local nonsatiation.

small distance away from x, denoted by $\varepsilon > 0$, there is another bundle $y \in \mathbb{R}^L_+$ within this distance from x that is preferred to x. Note that the bundle y may even have less of every commodity than x, as shown in the figure. Nonetheless, when $X = \mathbb{R}^L_+$ local nonsatiation rules out the extreme situation in which all commodities are bads, since in that case no consumption at all (the point $x = 0$) would be a satiation point.

Exercise 3.B.1: Show the following:

(a) If $\succsim$ is strongly monotone, then it is monotone.
(b) If $\succsim$ is monotone, then it is locally nonsatiated.

Given the preference relation $\succsim$ and a consumption bundle x, we can define three related sets of consumption bundles. The *indifference set* containing point x is the set of all bundles that are indifferent to x; formally, it is $\{y \in X: y \sim x\}$. The *upper contour set* of bundle x is the set of all bundles that are at least as good as x: $\{y \in X : y \succsim x\}$. The *lower contour set* of x is the set of all bundles that x is at least as good as: $\{y \in X : x \succsim y\}$.

One implication of local nonsatiation (and, hence, of monotonicity) is that it rules out "thick" indifference sets. The indifference set in Figure 3.B.2(a) cannot satisfy local nonsatiation because, if it did, there would be a better point than x within the circle drawn. In contrast, the indifference set in Figure 3.B.2(b) is compatible with local nonsatiation. Figure 3.B.2(b) also depicts the upper and lower contour sets of x.

(ii) *Convexity assumptions.* A second significant assumption, that of convexity of $\succsim$, concerns the trade-offs that the consumer is willing to make among different goods.

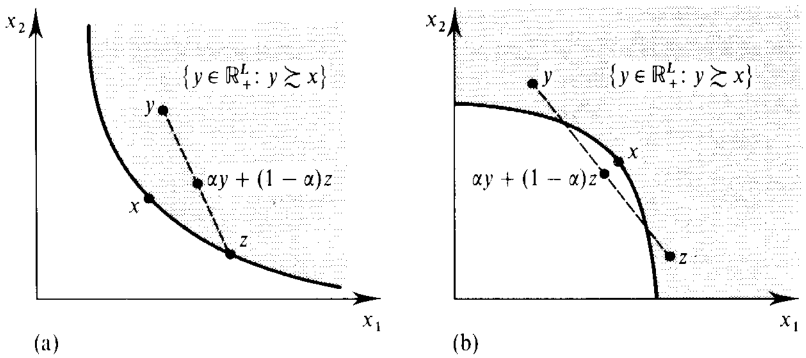


Figure 3.B.3
(a) Convex preferences.
(b) Nonconvex preferences.

Definition 3.B.4: The preference relation $\succsim$ on X is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

Figure 3.B.3(a) depicts a convex upper contour set; Figure 3.B.3(b) shows an upper contour set that is not convex.

Convexity is a strong but central hypothesis in economics. It can be interpreted in terms of *diminishing marginal rates of substitution*: That is, with convex preferences, from any initial consumption situation x, and for any two commodities, it takes increasingly larger amounts of one commodity to compensate for successive unit losses of the other.[4]

Convexity can also be viewed as the formal expression of a basic inclination of economic agents for diversification. Indeed, under convexity, if x is indifferent to y, then $\frac{1}{2}x + \frac{1}{2}y$, the half-half mixture of x and y, cannot be worse than either x or y. In Chapter 6, we shall give a diversification interpretation in terms of behavior under uncertainty. A taste for diversification is a realistic trait of economic life. Economic theory would be in serious difficulty if this postulated propensity for diversification did not have significant descriptive content. But there is no doubt that one can easily think of choice situations where it is violated. For example, you may like both milk and orange juice but get less pleasure from a mixture of the two.

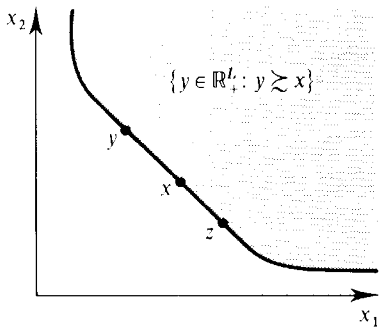
Definition 3.B.4 has been stated for a general consumption set X. But de facto, the convexity assumption can hold only if X is convex. Thus, the hypothesis rules out commodities being consumable only in integer amounts or situations such as that presented in Figure 2.C.3.

Although the convexity assumption on preferences may seem strong, this appearance should be qualified in two respects: First, a good number (although not all) of the results of this chapter extend without modification to the nonconvex case. Second, as we show in Appendix A of Chapter 4 and in Section 17.I, nonconvexities can often be incorporated into the theory by exploiting regularizing aggregation effects across consumers.

We also make use at times of a strengthening of the convexity assumption.

Definition 3.B.5: The preference relation $\succsim$ on X is *strictly convex* if for every x, we have that $y \succsim x$, $z \succsim x$, and $y \neq z$ implies $\alpha y + (1 - \alpha)z \succ x$ for all $\alpha \in (0, 1)$.

4. More generally, convexity is equivalent to a diminishing marginal rate of substitution between any two goods, provided that we allow for "composite commodities" formed from linear combinations of the L basic commodities.



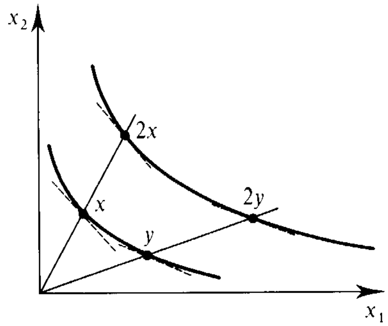


Figure 3.B.4 (left)
A convex, but not strictly convex, preference relation.

Figure 3.B.5 (right)
Homothetic preferences.

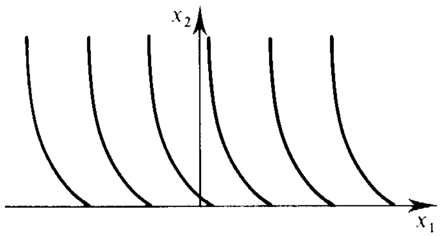


Figure 3.B.6
Quasilinear preferences.

Figure 3.B.3(a) showed strictly convex preferences. In Figure 3.B.4, on the other hand, the preferences, although convex, are not strictly convex.

In applications (particularly those of an econometric nature), it is common to focus on preferences for which it is possible to deduce the consumer's entire preference relation from a single indifference set. Two examples are the classes of *homothetic* and *quasilinear* preferences.

Definition 3.B.6: A monotone preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is *homothetic* if all indifference sets are related by proportional expansion along rays; that is, if $x \sim y$, then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$.

Figure 3.B.5 depicts a homothetic preference relation.

Definition 3.B.7: The preference relation $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}^{L-1}_+$ is *quasilinear* with respect to commodity 1 (called, in this case, the *numeraire* commodity) if[5]

(i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.

(ii) Good 1 is desirable; that is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Note that, in Definition 3.B.7, we assume that there is no lower bound on the possible consumption of the first commodity [the consumption set is $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$]. This assumption is convenient in the case of quasilinear preferences (Exercise 3.D.4 will illustate why). Figure 3.B.6 shows a quasilinear preference relation.

5. More generally, preferences can be quasilinear with respect to any commodity ℓ.

3.C Preference and Utility

For analytical purposes, it is very helpful if we can summarize the consumer's preferences by means of a utility function because mathematical programming techniques can then be used to solve the consumer's problem. In this section, we study when this can be done. Unfortunately, with the assumptions made so far, a rational preference relation need not be representable by a utility function. We begin with an example illustrating this fact and then introduce a weak, economically natural assumption (called *continuity*) that guarantees the existence of a utility representation.

Example 3.C.1: *The Lexicographic Preference Relation.* For simplicity, assume that $X = \mathbb{R}^2_+$. Define $x \succsim y$ if either "$x_1 > y_1$" or "$x_1 = y_1$ and $x_2 \geq y_2$." This is known as the *lexicographic preference relation.* The name derives from the way a dictionary is organized; that is, commodity 1 has the highest priority in determining the preference ordering, just as the first letter of a word does in the ordering of a dictionary. When the level of the first commodity in two commodity bundles is the same, the amount of the second commodity in the two bundles determines the consumer's preferences. In Exercise 3.C.1, you are asked to verify that the lexicographic ordering is complete, transitive, strongly monotone, and strictly convex. Nevertheless, it can be shown that no utility function exists that represents this preference ordering. This is intuitive. With this preference ordering, no two distinct bundles are indifferent; indifference sets are singletons. Therefore, we have two dimensions of distinct indifference sets. Yet, each of these indifference sets must be assigned, in an order-preserving way, a different utility number from the one-dimensional real line. In fact, a somewhat subtle argument is actually required to establish this claim rigorously. It is given, for the more advanced reader, in the following paragraph.

Suppose there is a utility function $u(\cdot)$. For every x_1, we can pick a rational number $r(x_1)$ such that $u(x_1, 2) > r(x_1) > u(x_1, 1)$. Note that because of the lexicographic character of preferences, $x_1 > x'_1$ implies $r(x_1) > r(x'_1)$ [since $r(x_1) > u(x_1, 1) > u(x'_1, 2) > r(x'_1)$]. Therefore, $r(\cdot)$ provides a one-to-one function from the set of real numbers (which is uncountable) to the set of rational numbers (which is countable). This is a mathematical impossibility. Therefore, we conclude that there can be no utility function representing these preferences.

■

The assumption that is needed to ensure the existence of a utility function is that the preference relation be continuous.

Definition 3.C.1: The preference relation $\succsim$ on X is *continuous* if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n, $x = \lim_{n\to\infty} x^n$, and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.

Continuity says that the consumer's preferences cannot exhibit "jumps," with, for example, the consumer preferring each element in sequence $\{x^n\}$ to the corresponding element in sequence $\{y^n\}$ but suddenly reversing her preference at the limiting points of these sequences x and y.

An equivalent way to state this notion of continuity is to say that for all x, the upper contour set $\{y \in X: y \succsim x\}$ and the lower contour set $\{y \in X: x \succsim y\}$ are both *closed*; that is, they include their boundaries. Definition 3.C.1 implies that for any sequence of points $\{y^n\}_{n=1}^{\infty}$ with $x \succsim y^n$ for all n and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$ (just let $x^n = x$ for all n). Hence, continuity as defined in Definition 3.C.1 implies that the lower contour set is closed; the same is implied for the upper contour set. The reverse argument, that closedness of the lower and upper contour sets implies that Definition 3.C.1 holds, is more advanced and is left as an exercise (Exercise 3.C.3).

Example 3.C.1 continued: Lexicographic preferences are not continuous. To see this, consider the sequence of bundles $x^n = (1/n, 0)$ and $y^n = (0, 1)$. For every n, we have $x^n \succ y^n$. But $\lim_{n\to\infty} y^n = (0, 1) \succ (0, 0) = \lim_{n\to\infty} x^n$. In words, as long as the first component of x is larger than that of y, x is preferred to y even if y_2 is much larger than x_2. But as soon as the first components become equal, only the second components are relevant, and so the preference ranking is reversed at the limit points of the sequence. ■

It turns out that the continuity of $\succsim$ is sufficient for the existence of a utility function representation. In fact, it guarantees the existence of a *continuous* utility function.

Proposition 3.C.1: Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function $u(x)$ that represents $\succsim$.

Proof: For the case of $X = \mathbb{R}^L_+$ and a monotone preference relation, there is a relatively simple and intuitive proof that we present here with the help of Figure 3.C.1.

Denote the diagonal ray in $\mathbb{R}^L_+$ (the locus of vectors with all L components equal) by Z. It will be convenient to let e designate the L-vector whose elements are all equal to 1. Then $\alpha e \in Z$ for all nonnegative scalars $\alpha \geq 0$.

Note that for every $x \in \mathbb{R}^L_+$, monotonicity implies that $x \succsim 0$. Also note that for any $\bar{\alpha}$ such that $\bar{\alpha}e \gg x$ (as drawn in the figure), we have $\bar{\alpha}e \succsim x$. Monotonicity and continuity can then be shown to imply that there is a unique value $\alpha(x) \in [0, \bar{\alpha}]$ such that $\alpha(x)e \sim x$.

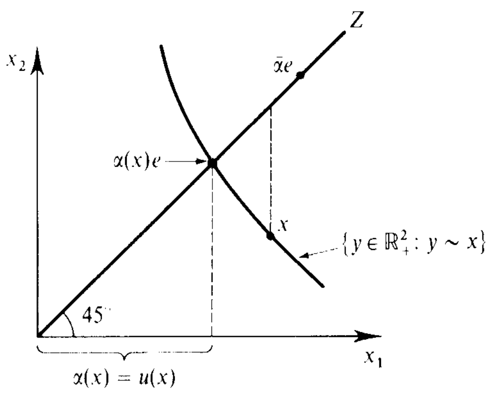


Figure 3.C.1
Construction of a utility function.

Formally, this can be shown as follows: By continuity, the upper and lower contour sets of x are closed. Hence, the sets $A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$ and $A^- = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ are nonempty and closed. Note that by completeness of $\succsim$, $\mathbb{R}_+ \subset (A^+ \cup A^-)$. The nonemptiness and closedness of A^+ and A^-, along with the fact that $\mathbb{R}_+$ is connected, imply that $A^+ \cap A^- \neq \varnothing$. Thus, there exists a scalar α such that $\alpha e \sim x$. Furthermore, by monotonicity, $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Hence, there can be at most one scalar satisfying $\alpha e \sim x$. This scalar is $\alpha(x)$.

We now take $\alpha(x)$ as our utility function; that is, we assign a utility value $u(x) = \alpha(x)$ to every x. This utility level is also depicted in Figure 3.C.1. We need to check two properties of this function: that it represents the preference $\succsim$ [i.e., that $\alpha(x) \geq \alpha(y) \Leftrightarrow x \succsim y$] and that it is a continuous function. The latter argument is more advanced, and therefore we present it in small type.

That $\alpha(x)$ represents preferences follows from its construction. Formally, suppose first that $\alpha(x) \geq \alpha(y)$. By monotonicity, this implies that $\alpha(x)e \succsim \alpha(y)e$. Since $x \sim \alpha(x)e$ and $y \sim \alpha(y)e$, we have $x \succsim y$. Suppose, on the other hand, that $x \succsim y$. Then $\alpha(x)e \sim x \succsim y \sim \alpha(y)e$; and so by monotonicity, we must have $\alpha(x) \geq \alpha(y)$. Hence, $\alpha(x) \geq \alpha(y) \Leftrightarrow x \succsim y$.

We now argue that $\alpha(x)$ is a continuous function at all x; that is, for any sequence $\{x^n\}_{n=1}^{\infty}$ with $x = \lim_{n\to\infty} x^n$, we have $\lim_{n\to\infty} \alpha(x^n) = \alpha(x)$. Hence, consider a sequence $\{x^n\}_{n=1}^{\infty}$ such that $x = \lim_{n\to\infty} x^n$.

We note first that the sequence $\{\alpha(x^n)\}_{n=1}^{\infty}$ must have a convergent subsequence. By monotonicity, for any $\varepsilon > 0$, $\alpha(x')$ lies in a compact subset of $\mathbb{R}_+$, $[\alpha_0, \alpha_1]$, for all x' such that $\|x' - x\| \leq \varepsilon$ (see Figure 3.C.2). Since $\{x^n\}_{n=1}^{\infty}$ converges to x, there exists an N such that $\alpha(x^n)$

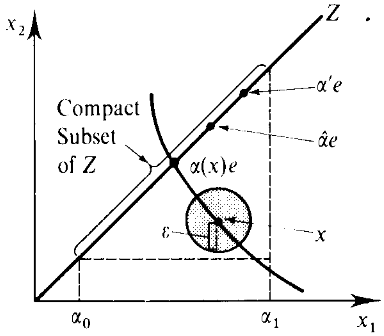


Figure 3.C.2
Proof that the constructed utility function is continuous.

lies in this compact set for all $n > N$. But any infinite sequence that lies in a compact set must have a convergent subsequence (see Section M.F of the Mathematical Appendix).

What remains is to establish that all convergent subsequences of $\{\alpha(x^n)\}_{n=1}^{\infty}$ converge to $\alpha(x)$. To see this, suppose otherwise: that there is some strictly increasing function $m(\cdot)$ that assigns to each positive integer n a positive integer $m(n)$ and for which the subsequence $\{\alpha(x^{m(n)})\}_{n=1}^{\infty}$ converges to $\alpha' \neq \alpha(x)$. We first show that $\alpha' > \alpha(x)$ leads to a contradiction. To begin, note that monotonicity would then imply that $\alpha'e \succ \alpha(x)e$. Now, let $\hat{\alpha} = \frac{1}{2}[\alpha' + \alpha(x)]$. The point $\hat{\alpha}e$ is the midpoint on Z between $\alpha'e$ and $\alpha(x)e$ (see Figure 3.C.2). By monotonicity, $\hat{\alpha}e \succ \alpha(x)e$. Now, since $\alpha(x^{m(n)}) \to \alpha' > \hat{\alpha}$, there exists an $\bar{N}$ such that for all $n > \bar{N}$, $\alpha(x^{m(n)}) > \hat{\alpha}$.

Hence, for all such n, $x^{m(n)} \sim \alpha(x^{m(n)})e \succ \hat{\alpha}e$ (where the latter relation follows from monotonicity). Because preferences are continuous, this would imply that $x \succsim \hat{\alpha}e$. But since $x \sim \alpha(x)e$, we get $\alpha(x)e \succsim \hat{\alpha}e$, which is a contradiction. The argument ruling out $\alpha' < \alpha(x)$ is similar. Thus, since all convergent subsequences of $\{\alpha(x^n)\}_{n=1}^{\infty}$ must converge to $\alpha(x)$, we have $\lim_{n \to \infty} \alpha(x^n) = \alpha(x)$, and we are done.

■

From now on, we assume that the consumer's preference relation is continuous and hence representable by a continuous utility function. As we noted in Section 1.B, the utility function $u(\cdot)$ that represents a preference relation $\succsim$ is not unique; any strictly increasing transformation of $u(\cdot)$, say $v(x) = f(u(x))$, where $f(\cdot)$ is a strictly increasing function, also represents $\succsim$. Proposition 3.C.1 tells us that if $\succsim$ is continuous, there exists *some* continuous utility function representing $\succsim$. But not all utility functions representing $\succsim$ are continuous; any strictly increasing but discontinuous transformation of a continuous utility function also represents $\succsim$.

For analytical purposes, it is also convenient if $u(\cdot)$ can be assumed to be differentiable. It is possible, however, for continuous preferences *not* to be representable by a differentiable utility function. The simplest example, shown in Figure 3.C.3, is the case of *Leontief* preferences, where $x'' \succsim x'$ if and only if $\text{Min}\{x''_1, x''_2\} \geq \text{Min}\{x'_1, x'_2\}$. The nondifferentiability arises because of the kink in indifference curves when $x_1 = x_2$.

Whenever convenient in the discussion that follows, we nevertheless assume utility functions to be twice continuously differentiable. It is possible to give a condition purely in terms of preferences that implies this property, but we shall not do so here. Intuitively, what is required is that indifference sets be smooth surfaces that fit together nicely so that the rates at which commodities substitute for each other depend differentiably on the consumption levels.

Restrictions on preferences translate into restrictions on the form of utility functions. The property of monotonicity, for example, implies that the utility function is increasing: $u(x) > u(y)$ if $x \gg y$.

The property of convexity of preferences, on the other hand, implies that $u(\cdot)$ is *quasiconcave* [and, similarly, strict convexity of preferences implies strict quasiconcavity of $u(\cdot)$]. The utility function $u(\cdot)$ is quasiconcave if the set $\{y \in \mathbb{R}^L_+ : u(y) \geq u(x)\}$ is convex for all x or, equivalently, if $u(\alpha x + (1 - \alpha)y) \geq \text{Min}\{u(x), u(y)\}$ for

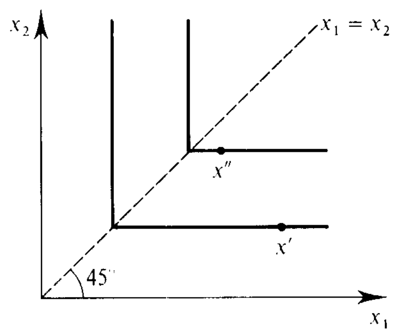


Figure 3.C.3
Leontief preferences cannot be represented by a differentiable utility function.

any x, y and all $\alpha \in [0, 1]$. [If the inequality is strict for all $x \neq y$ and $\alpha \in (0, 1)$ then $u(\cdot)$ is strictly quasiconcave; for more on quasiconcavity and strict quasiconcavity see Section M.C of the Mathematical Appendix.] Note, however, that convexity of $\succsim$ does *not* imply the stronger property that $u(\cdot)$ is concave [that $u(\alpha x + (1 - \alpha)y) \geq \alpha u(x) + (1 - \alpha)u(y)$ for any x, y and all $\alpha \in [0, 1]$]. In fact, although this is a somewhat fine point, there may not be *any* concave utility function representing a particular convex preference relation $\succsim$.

In Exercise 3.C.5, you are asked to prove two other results relating utility representations and underlying preference relations:

(i) A continuous $\succsim$ on $X = \mathbb{R}^L_+$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one [i.e., such that $u(\alpha x) = \alpha u(x)$ for all $\alpha > 0$].

(ii) A continuous $\succsim$ on $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$ is quasilinear with respect to the first commodity if and only if it admits a utility function $u(x)$ of the form $u(x) = x_1 + \phi(x_2, \ldots, x_L)$.

It is important to realize that although monotonicity and convexity of $\succsim$ imply that *all* utility functions representing $\succsim$ are increasing and quasiconcave, (i) and (ii) merely say that there is at *least one* utility function that has the specified form. Increasingness and quasiconcavity are *ordinal* properties of $u(\cdot)$; they are preserved for any arbitrary increasing transformation of the utility index. In contrast, the special forms of the utility representations in (i) and (ii) are not preserved; they are *cardinal* properties that are simply convenient choices for a utility representation.[6]

3.D The Utility Maximization Problem

We now turn to the study of the consumer's decision problem. We assume throughout that the consumer has a rational, continuous, and locally nonsatiated preference relation, and we take $u(x)$ to be a continuous utility function representing these preferences. For the sake of concreteness, we also assume throughout the remainder of the chapter that the consumption set is $X = \mathbb{R}^L_+$.

The consumer's problem of choosing her most preferred consumption bundle given prices $p \gg 0$ and wealth level $w > 0$ can now be stated as the following *utility maximization problem (UMP)*:

$$\begin{aligned} \underset{x \geq 0}{\text{Max}} \quad & u(x) \\ \text{s.t.} \quad & p \cdot x \leq w. \end{aligned}$$

In the UMP, the consumer chooses a consumption bundle in the Walrasian budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ to maximize her utility level. We begin with the results stated in Proposition 3.D.1.

Proposition 3.D.1: If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

6. Thus, in this sense, continuity is also a cardinal property of utility functions. See also the discussion of ordinal and cardinal properties of utility representations in Section 1.B.

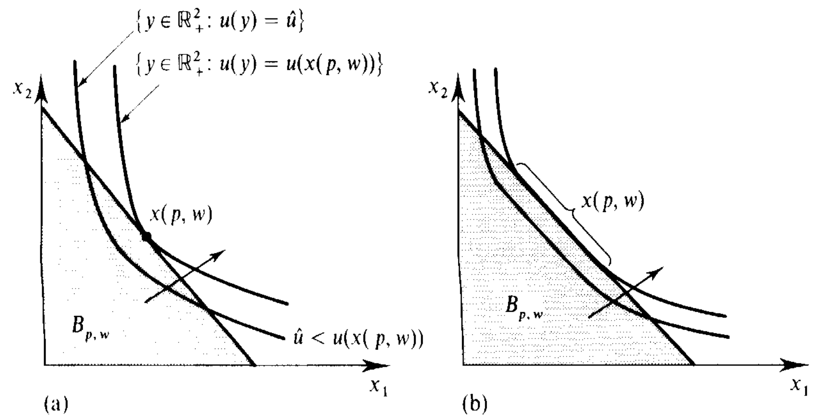


Figure 3.D.1
The utility maximization problem (UMP).
(a) Single solution.
(b) Multiple solutions.

Proof: If $p \gg 0$, then the budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is a compact set because it is both bounded [for any $\ell = 1, \ldots, L$, we have $x_\ell \leq (w/p_\ell)$ for all $x \in B_{p,w}$] and closed. The result follows from the fact that a continuous function always has a maximum value on any compact set (set Section M.F. of the Mathematical Appendix). ■

With this result, we now focus our attention on the properties of two objects that emerge from the UMP: the consumer's set of optimal consumption bundles (the solution set of the UMP) and the consumer's maximal utility value (the value function of the UMP).

The Walrasian Demand Correspondence/Function

The rule that assigns the set of optimal consumption vectors in the UMP to each price-wealth situation $(p, w) \gg 0$ is denoted by $x(p, w) \in \mathbb{R}^L_+$ and is known as the *Walrasian* (or *ordinary* or *market*) *demand correspondence*. An example for $L = 2$ is depicted in Figure 3.D.1(a), where the point $x(p, w)$ lies in the indifference set with the highest utility level of any point in $B_{p,w}$. Note that, as a general matter, for a given $(p, w) \gg 0$ the optimal set $x(p, w)$ may have more than one element, as shown in Figure 3.D.1(b). When $x(p, w)$ is single-valued for all (p, w), we refer to it as the *Walrasian* (or *ordinary* or *market*) *demand function.*[7]

The properties of $x(p, w)$ stated in Proposition 3.D.2 follow from direct examination of the UMP.

Proposition 3.D.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:

7. This demand function has also been called the *Marshallian demand function*. However, this terminology can create confusion, and so we do not use it here. In Marshallian partial equilibrium analysis (where wealth effects are absent), all the different kinds of demand functions studied in this chapter coincide, and so it is not clear which of these demand functions would deserve the Marshall name in the more general setting.

(i) *Homogeneity of degree zero in* (p, w): $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar $\alpha > 0$.

(ii) *Walras' law:* $p \cdot x = w$ for all $x \in x(p, w)$.

(iii) *Convexity/uniqueness:* If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if $\succsim$ is *strictly convex,* so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Proof: We establish each of these properties in turn.

(i) For homogeneity, note that for any scalar $\alpha > 0$,

$$\{x \in \mathbb{R}^L_+ : \alpha p \cdot x \leq \alpha w\} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\};$$

that is, the set of feasible consumption bundles in the UMP does not change when all prices and wealth are multiplied by a constant $\alpha > 0$. The set of utility-maximizing consumption bundles must therefore be the same in these two circumstances, and so $x(p, w) = x(\alpha p, \alpha w)$. Note that this property does not require any assumptions on $u(\cdot)$.

(ii) Walras' law follows from local nonsatiation. If $p \cdot x < w$ for some $x \in x(p, w)$, then there must exist another consumption bundle y sufficiently close to x with both $p \cdot y < w$ and $y \succ x$ (see Figure 3.D.2). But this would contradict x being optimal in the UMP.

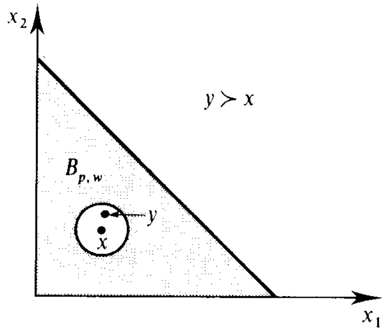


Figure 3.D.2
Local nonsatiation implies Walras' law.

(iii) Suppose that $u(\cdot)$ is quasiconcave and that there are two bundles x and x', with $x \neq x'$, both of which are elements of $x(p, w)$. To establish the result, we show that $x'' = \alpha x + (1 - \alpha)x'$ is an element of $x(p, w)$ for any $\alpha \in [0,1]$. To start, we know that $u(x) = u(x')$. Denote this utility level by u^*. By quasiconcavity, $u(x'') \geq u^*$ [see Figure 3.D.3(a)]. In addition, since $p \cdot x \leq w$ and $p \cdot x' \leq w$, we also have

$$p \cdot x'' = p \cdot [\alpha x + (1 - \alpha)x'] \leq w.$$

Therefore, x'' is a feasible choice in the UMP (put simply, x'' is feasible because $B_{p,w}$ is a convex set). Thus, since $u(x'') \geq u^*$ and x'' is feasible, we have $x'' \in x(p, w)$. This establishes that $x(p, w)$ is a convex set if $u(\cdot)$ is quasiconcave.

Suppose now that $u(\cdot)$ is *strictly* quasiconcave. Following the same argument but using strict quasiconcavity, we can establish that x'' is a feasible choice and that $u(x'') > u^*$ for all $\alpha \in (0,1)$. Because this contradicts the assumption that x and x' are elements of $x(p, w)$, we conclude that there can be at most one element in $x(p, w)$. Figure 3.D.3(b) illustrates this argument. Note the difference from Figure 3.D.3(a) arising from the strict quasiconcavity of $u(x)$. ■

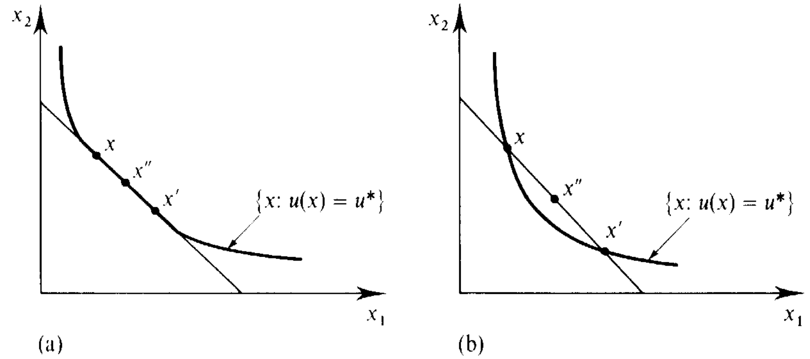


Figure 3.D.3
(a) Convexity of preferences implies convexity of $x(p, w)$.
(b) Strict convexity of preferences implies that $x(p, w)$ is single-valued.

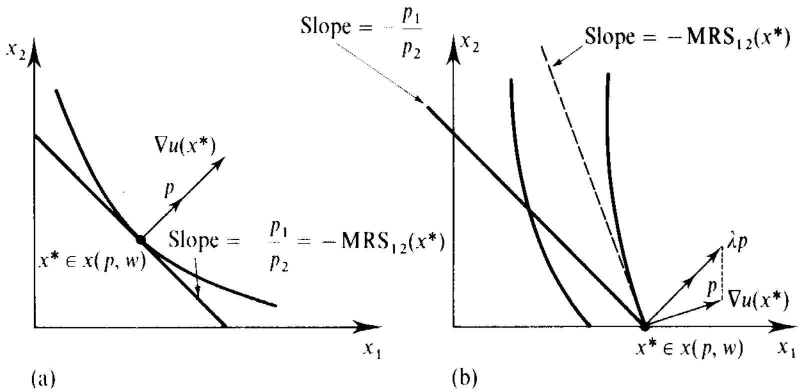


Figure 3.D.4
(a) Interior solution.
(b) Boundary solution.

If $u(\cdot)$ is continuously differentiable, an optimal consumption bundle $x^* \in x(p, w)$ can be characterized in a very useful manner by means of first-order conditions. The *Kuhn Tucker (necessary) conditions* (see Section M.K of the Mathematical Appendix) say that if $x^* \in x(p, w)$ is a solution to the UMP, then there exists a *Lagrange multiplier* $\lambda \geq 0$ such that for all $\ell = 1, \ldots, L$:[8]

$$\frac{\partial u(x^*)}{\partial x_\ell} \leq \lambda p_\ell, \quad \text{with equality if } x_\ell^* > 0. \tag{3.D.1}$$

Equivalently, if we let $\nabla u(x) = [\partial u(x)/\partial x_1, \ldots, \partial u(x)/\partial x_L]$ denote the gradient vector of $u(\cdot)$ at x, we can write (3.D.1) in matrix notation as

$$\nabla u(x^*) \leq \lambda p \tag{3.D.2}$$

and

$$x^* \cdot [\nabla u(x^*) - \lambda p] = 0. \tag{3.D.3}$$

Thus, if we are at an interior optimum (i.e., if $x^* \gg 0$), we must have

$$\nabla u(x^*) = \lambda p. \tag{3.D.4}$$

Figure 3.D.4(a) depicts the first-order conditions for the case of an interior optimum when $L = 2$. Condition (3.D.4) tells us that at an interior optimum, the

8. To be fully rigorous, these Kuhn Tucker necessary conditions are valid only if the constraint qualification condition holds (see Section M.K of the Mathematical Appendix). In the UMP, this is always so. Whenever we use Kuhn Tucker necessary conditions without mentioning the constraint qualification condition, this requirement is met.

gradient vector of the consumer's utility function $\nabla u(x^*)$ must be proportional to the price vector p, as is shown in Figure 3.D.4(a). If $\nabla u(x^*) \gg 0$, this is equivalent to the requirement that for any two goods ℓ and k, we have

$$\frac{\partial u(x^*)/\partial x_\ell}{\partial u(x^*)/\partial x_k} = \frac{p_\ell}{p_k}. \tag{3.D.5}$$

The expression on the left of (3.D.5) is the *marginal rate of substitution of good ℓ for good k at x^*, $MRS_{\ell k}(x^*)$*; it tells us the amount of good k that the consumer must be given to compensate her for a one-unit marginal reduction in her consumption of good ℓ.[9] In the case where $L = 2$, the slope of the consumer's indifference set at x^* is precisely $-MRS_{12}(x^*)$. Condition (3.D.5) tells us that at an interior optimum, the consumer's marginal rate of substitution between any two goods must be equal to their price ratio, the marginal rate of exchange between them, as depicted in Figure 3.D.4(a). Were this not the case, the consumer could do better by marginally changing her consumption. For example, if $[\partial u(x^*)/\partial x_\ell]/[\partial u(x^*)/\partial x_k] > (p_\ell/p_k)$, then an increase in the consumption of good ℓ of dx_ℓ, combined with a decrease in good k's consumption equal to $(p_\ell/p_k)\,dx_\ell$, would be feasible and would yield a utility change of $[\partial u(x^*)/\partial x_\ell]\,dx_\ell - [\partial u(x^*)/\partial x_k](p_\ell/p_k)\,dx_\ell > 0$.

Figure 3.D.4(b) depicts the first-order conditions for the case of $L = 2$ when the consumer's optimal bundle x^* lies on the boundary of the consumption set (we have $x_2^* = 0$ there). In this case, the gradient vector need not be proportional to the price vector. In particular, the first-order conditions tell us that $\partial u_\ell(x^*)/\partial x_\ell \le \lambda p_\ell$ for those ℓ with $x_\ell^* = 0$ and $\partial u_\ell(x^*)/\partial x_\ell = \lambda p_\ell$ for those ℓ with $x_\ell^* > 0$. Thus, in the figure, we see that $MRS_{12}(x^*) > p_1/p_2$. In contrast with the case of an interior optimum, an inequality between the marginal rate of substitution and the price ratio can arise at a boundary optimum because the consumer is unable to reduce her consumption of good 2 (and correspondingly increase her consumption of good 1) any further.

The Lagrange multiplier λ in the first-order conditions (3.D.2) and (3.D.3) gives the marginal, or shadow, value of relaxing the constraint in the UMP (this is a general property of Lagrange multipliers; see Sections M.K and M.L of the Mathematical Appendix). It therefore equals the consumer's marginal utility value of wealth at the optimum. To see this directly, consider for simplicity the case where $x(p, w)$ is a differentiable function and $x(p, w) \gg 0$. By the chain rule, the change in utility from a marginal increase in w is given by $\nabla u(x(p, w)) \cdot D_w x(p, w)$, where $D_w x(p, w) = [\partial x_1(p, w)/\partial w, \ldots, \partial x_L(p, w)/\partial w]$. Substituting for $\nabla u(x(p, w))$ from condition (3.D.4), we get

$$\nabla u(x(p, w)) \cdot D_w x(p, w) = \lambda p \cdot D_w x(p, w) = \lambda,$$

where the last equality follows because $p \cdot x(p, w) = w$ holds for all w (Walras' law) and therefore $p \cdot D_w x(p, w) = 1$. Thus, the marginal change in utility arising from

9. Note that if utility is unchanged with differential changes in x_ℓ and x_k, dx_ℓ and dx_k, then $[\partial u(x)/\partial x_\ell]\,dx_\ell + [\partial u(x)/\partial x_k]\,dx_k = 0$. Thus, when x_ℓ falls by amount $dx_\ell < 0$, the increase required in x_k to keep utility unchanged is precisely $dx_k = MRS_{\ell k}(x^*)(-dx_\ell)$.

a marginal increase in wealth—the consumer's *marginal utility of wealth*—is precisely λ.[10]

We have seen that conditions (3.D.2) and (3.D.3) must necessarily be satisfied by any $x^* \in x(p, w)$. When, on the other hand, does satisfaction of these first-order conditions by some bundle x imply that x is a solution to the UMP? That is, when are the first-order conditions *sufficient* to establish that x is a solution? If $u(\cdot)$ is quasiconcave and monotone and has $\nabla u(x) \neq 0$ for all $x \in \mathbb{R}^L_+$, then the Kuhn–Tucker first-order conditions are indeed sufficient (see Section M.K of the Mathematical Appendix). What if $u(\cdot)$ is not quasiconcave? In that case, if $u(\cdot)$ is locally quasiconcave at x^*, and if x^* satisfies the first-order conditions, then x^* is a local maximum. Local quasiconcavity can be verified by means of a determinant test on the *bordered Hessian matrix* of $u(\cdot)$ at x^*. (For more on this, see Sections M.C and M.D of the Mathematical Appendix.)

Example 3.D.1 illustrates the use of the first-order conditions in deriving the consumer's optimal consumption bundle.

Example 3.D.1: *The Demand Function Derived from the Cobb–Douglas Utility Function.* A *Cobb–Douglas* utility function for $L = 2$ is given by $u(x_1, x_2) = kx_1^{\alpha}x_2^{1-\alpha}$ for some $\alpha \in (0, 1)$ and $k > 0$. It is increasing at all $(x_1, x_2) \gg 0$ and is homogeneous of degree one. For our analysis, it turns out to be easier to use the increasing transformation $\alpha \ln x_1 + (1 - \alpha) \ln x_2$, a strictly concave function, as our utility function. With this choice, the UMP can be stated as

$$\underset{x_1, x_2}{\text{Max}} \quad \alpha \ln x_1 + (1 - \alpha) \ln x_2 \qquad (3.D.6)$$
$$\text{s.t. } p_1x_1 + p_2x_2 = w.$$

[Note that since $u(\cdot)$ is increasing, the budget constraint will hold with strict equality at any solution.]

Since $\ln 0 = -\infty$, the optimal choice $(x_1(p, w), x_2(p, w))$ is strictly positive and must satisfy the first-order conditions (we write the consumption levels simply as x_1 and x_2 for notational convenience)

$$\frac{\alpha}{x_1} = \lambda p_1 \qquad (3.D.7)$$

and

$$\frac{1 - \alpha}{x_2} = \lambda p_2 \qquad (3.D.8)$$

for some $\lambda \geq 0$, and the budget constraint $p \cdot x(p, w) = w$. Conditions (3.D.7) and (3.D.8) imply that

$$p_1x_1 = \frac{\alpha}{1 - \alpha} p_2x_2$$

or, using the budget constraint,

$$p_1x_1 = \frac{\alpha}{1 - \alpha}(w - p_1x_1).$$

10. Note that if monotonicity of $u(\cdot)$ is strengthened slightly by requiring that $\nabla u(x) \geq 0$ *and* $\nabla u(x) \neq 0$ for all x, then condition (3.D.4) and $p \gg 0$ also imply that λ is strictly positive at any solution of the UMP.

Hence (including the arguments of x_1 and x_2 once again)

$$x_1(p,w) = \frac{\alpha w}{p_1},$$

and (using the budget constraint)

$$x_2(p,w) = \frac{(1-\alpha)w}{p_2}.$$

Note that with the Cobb–Douglas utility function, the expenditure on each commodity is a constant fraction of wealth for any price vector p [a share of α goes for the first commodity and a share of $(1-\alpha)$ goes for the second]. ■

Exercise 3.D.1: Verify the three properties of Proposition 3.D.2 for the Walrasian demand function generated by the Cobb–Douglas utility function.

For the analysis of demand responses to changes in prices and wealth, it is also very helpful if the consumer's Walrasian demand is suitably continuous and differentiable. Because the issues are somewhat more technical, we will discuss the conditions under which demand satisfies these properties in Appendix A to this chapter. We conclude there that both properties hold under fairly general conditions. Indeed, if preferences are continuous, strictly convex, and locally nonsatiated on the consumption set $\mathbb{R}^L_+$, then $x(p, w)$ (which is then a function) is *always* continuous at all $(p, w) \gg 0$.

The Indirect Utility Function

For each $(p, w) \gg 0$, the utility value of the UMP is denoted $v(p, w) \in \mathbb{R}$. It is equal to $u(x^*)$ for any $x^* \in x(p, w)$. The function $v(p, w)$ is called the *indirect utility function* and often proves to be a very useful analytic tool. Proposition 3.D.3 identifies its basic properties.

Proposition 3.D.3: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The indirect utility function $v(p, w)$ is

(i) Homogeneous of degree zero.

(ii) Strictly increasing in w and nonincreasing in p_ℓ for any ℓ.

(iii) Quasiconvex; that is, the set $\{(p, w): v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.[11]

(iv) Continuous in p and w.

Proof: Except for quasiconvexity and continuity all the properties follow readily from our previous discussion. We forgo the proof of continuity here but note that, when preferences are strictly convex, it follows from the fact that $x(p, w)$ and $u(x)$ are continuous functions because $v(p, w) = u(x(p, w))$ [recall that the continuity of $x(p, w)$ is established in Appendix A of this chapter].

To see that $v(p,w)$ is quasiconvex, suppose that $v(p, w) \leq \bar{v}$ and $v(p', w') \leq \bar{v}$. For any $\alpha \in [0, 1]$, consider then the price–wealth pair $(p'', w'') = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$.

11. Note that property (iii) says that $v(p, w)$ is quasi*convex*, *not* quasi*concave*. Observe also that property (iii) does not require for its validity that $u(\cdot)$ be quasiconcave.

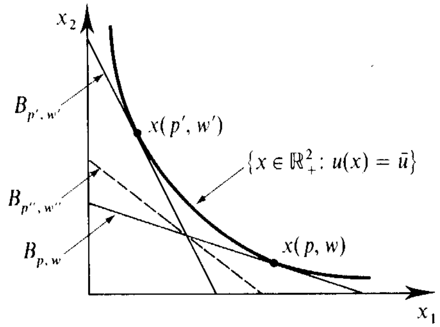


Figure 3.D.5
The indirect utility function $v(p, w)$ is quasiconvex.

To establish quasiconvexity, we want to show that $v(p'', w'') \le \bar{v}$. Thus, we show that for any x with $p'' \cdot x \le w''$, we must have $u(x) \le \bar{v}$. Note, first, that if $p'' \cdot x \le w''$, then,

$$\alpha p \cdot x + (1 - \alpha) p' \cdot x \le \alpha w + (1 - \alpha) w'.$$

Hence, either $p \cdot x \le w$ or $p' \cdot x \le w'$ (or both). If the former inequality holds, then $u(x) \le v(p, w) \le v$, and we have established the result. If the latter holds, then $u(x) \le v(p', w') \le v$, and the same conclusion follows. ■

The quasiconvexity of $v(p, w)$ can be verified graphically in Figure 3.D.5 for the case where $L = 2$. There, the budget sets for price–wealth pairs (p, w) and (p', w') generate the same maximized utility value $\bar{u}$. The budget line corresponding to $(p'', w'') = (\alpha p + (1 - \alpha) p', \alpha w + (1 - \alpha) w')$ is depicted as a dashed line in Figure 3.D.5. Because (p'', w'') is a convex combination of (p, w) and (p', w'), its budget line lies between the budget lines for these two price–wealth pairs. As can be seen in the figure, the attainable utility under (p'', w'') is necessarily no greater than $\bar{u}$.

Note that the indirect utility function depends on the utility representation chosen. In particular, if $v(p, w)$ is the indirect utility function when the consumer's utility function is $u(\cdot)$, then the indirect utility function corresponding to utility representation $\tilde{u}(x) = f(u(x))$ is $\tilde{v}(p, w) = f(v(p, w))$.

Example 3.D.2: Suppose that we have the utility function $u(x_1, x_2) = \alpha \ln x_1 + (1 - \alpha) \ln x_2$. Then, substituting $x_1(p, w)$ and $x_2(p, w)$ from Example 3.D.1, into $u(x)$ we have

$$\begin{aligned} v(p, w) &= u(x(p, w)) \\ &= [\alpha \ln \alpha + (1 - \alpha) \ln (1 - \alpha)] + \ln w - \alpha \ln p_1 - (1 - \alpha) \ln p_2. \end{aligned}$$

Exercise 3.D.2: Verify the four properties of Proposition 3.D.3 for the indirect utility function derived in Example 3.D.2.

3.E The Expenditure Minimization Problem

In this section, we study the following *expenditure minimization problem* (EMP) for $p \gg 0$ and $u > u(0)$:[12]

12. Utility $u(0)$ is the utility from consuming the consumption bundle $x = (0, 0, \ldots, 0)$. The restriction to $u > u(0)$ rules out only uninteresting situations.

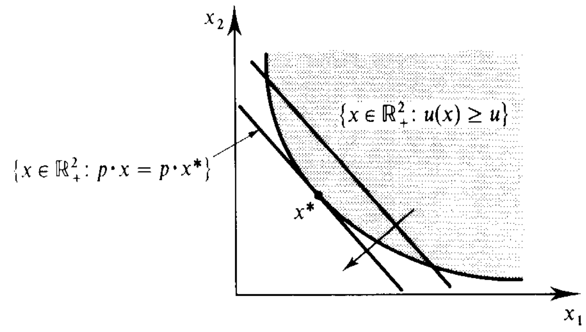


Figure 3.E.1
The expenditure minimization problem (EMP).

$$\underset{x \geq 0}{\text{Min}} \quad p \cdot x \qquad \text{(EMP)}$$
$$\text{s.t. } u(x) \geq u.$$

Whereas the UMP computes the maximal level of utility that can be obtained given wealth w, the EMP computes the minimal level of wealth required to reach utility level u. The EMP is the "dual" problem to the UMP. It captures the same aim of efficient use of the consumer's purchasing power while reversing the roles of objective function and constraint.[13]

Throughout this section, we assume that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $\mathbb{R}^L_+$.

The EMP is illustrated in Figure 3.E.1. The optimal consumption bundle x^* is the least costly bundle that still allows the consumer to achieve the utility level u. Geometrically, it is the point in the set $\{x \in \mathbb{R}^L_+: u(x) \geq u\}$ that lies on the lowest possible budget line associated with the price vector p.

Proposition 3.E.1 describes the formal relationship between EMP and the UMP.

Proposition 3.E.1: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have

(i) If x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w.

(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this UMP is exactly u.

Proof: (i) Suppose that x^* is not optimal in the EMP with required utility level $u(x^*)$. Then there exists an x' such that $u(x') \geq u(x^*)$ and $p \cdot x' < p \cdot x^* \leq w$. By local nonsatiation, we can find an x'' very close to x' such that $u(x'') > u(x')$ and $p \cdot x'' < w$. But this implies that $x'' \in B_{p,w}$ and $u(x'') > u(x^*)$, contradicting the optimality of x^* in the UMP. Thus, x^* must be optimal in the EMP when the required utility level

13. The term "dual" is meant to be suggestive. It is usually applied to pairs of problems and concepts that are formally similar except that the role of quantities and prices, and/or maximization and minimization, and/or objective function and constraint, have been reversed.

is $u(x^*)$, and the minimized expenditure level is therefore $p \cdot x^*$. Finally, since x^* solves the UMP when wealth is w, by Walras' law we have $p \cdot x^* = w$.

(ii) Since $u > u(0)$, we must have $x^* \neq 0$. Hence, $p \cdot x^* > 0$. Suppose that x^* is not optimal in the UMP when wealth is $p \cdot x^*$. Then there exists an x' such that $u(x') > u(x^*)$ and $p \cdot x' \leq p \cdot x^*$. Consider a bundle $x'' = \alpha x'$ where $\alpha \in (0,1)$ (x'' is a "scaled-down" version of x'). By continuity of $u(\cdot)$, if α is close enough to 1, then we will have $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$. But this contradicts the optimality of x^* in the EMP. Thus, x^* must be optimal in the UMP when wealth is $p \cdot x^*$, and the maximized utility level is therefore $u(x^*)$. In Proposition 3.E.3(ii), we will show that if x^* solves the EMP when the required utility level is u, then $u(x^*) = u$. ■

As with the UMP, when $p \gg 0$ a solution to the EMP exists under very general conditions. The constraint set merely needs to be nonempty; that is, $u(\cdot)$ must attain values at least as large as u for *some* x (see Exercise 3.E.3). From now on, we assume that this is so; for example, this condition will be satisfied for any $u > u(0)$ if $u(\cdot)$ is unbounded above.

We now proceed to study the optimal consumption vector and the value function of the EMP. We consider the value function first.

The Expenditure Function

Given prices $p \gg 0$ and required utility level $u > u(0)$, the value of the EMP is denoted $e(p, u)$. The function $e(p, u)$ is called the *expenditure function.* Its value for any (p, u) is simply $p \cdot x^*$, where x^* is any solution to the EMP. The result in Proposition 3.E.2 describes the basic properties of the expenditure function. It parallels Proposition 3.D.3's characterization of the properties of the indirect utility function for the UMP.

Proposition 3.E.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The expenditure function $e(p, u)$ is

(i) Homogeneous of degree one in p.
(ii) Strictly increasing in u and nondecreasing in p_ℓ for any ℓ.
(iii) Concave in p.
(iv) Continuous in p and u.

Proof: We prove only properties (i), (ii), and (iii).

(i) The constraint set of the EMP is unchanged when prices change. Thus, for any scalar $\alpha > 0$, minimizing $(\alpha p) \cdot x$ on this set leads to the same optimal consumption bundles as minimizing $p \cdot x$. Letting x^* be optimal in both circumstances, we have $e(\alpha p, u) = \alpha p \cdot x^* = \alpha e(p, u)$.

(ii) Suppose that $e(p, u)$ were not strictly increasing in u, and let x' and x'' denote optimal consumption bundles for required utility levels u' and u'', respectively, where $u'' > u'$ and $p \cdot x' \geq p \cdot x'' > 0$. Consider a bundle $\tilde{x} = \alpha x''$, where $\alpha \in (0, 1)$. By continuity of $u(\cdot)$, there exists an α close enough to 1 such that $u(\tilde{x}) > u'$ and $p \cdot x' > p \cdot \tilde{x}$. But this contradicts x' being optimal in the EMP with required utility level u'.

To show that $e(p, u)$ is nondecreasing in p_ℓ, suppose that price vectors p'' and p' have $p''_\ell \geq p'_\ell$ and $p''_k = p'_k$ for all $k \neq \ell$. Let x'' be an optimizing vector in the EMP for prices p''. Then $e(p'', u) = p'' \cdot x'' \geq p' \cdot x'' \geq e(p', u)$, where the latter inequality follows from the definition of $e(p', u)$.

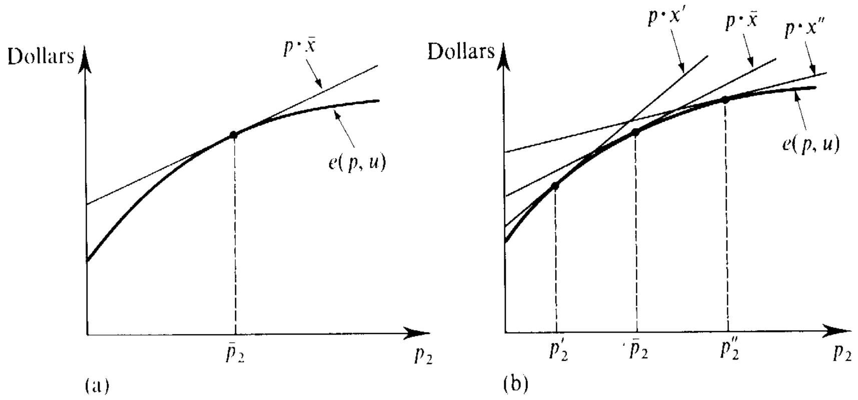


Figure 3.E.2
The concavity in p of the expenditure function.

(iii) For concavity, fix a required utility level $\bar{u}$, and let $p'' = \alpha p + (1 - \alpha)p'$ for $\alpha \in [0, 1]$. Suppose that x'' is an optimal bundle in the EMP when prices are p''. If so,

$$\begin{aligned} e(p'', \bar{u}) &= p'' \cdot x'' \\ &= \alpha p \cdot x'' + (1 - \alpha)p' \cdot x'' \\ &\geq \alpha e(p, \bar{u}) + (1 - \alpha)e(p', \bar{u}), \end{aligned}$$

where the last inequality follows because $u(x'') \geq \bar{u}$ and the definition of the expenditure function imply that $p \cdot x'' \geq e(p, \bar{u})$ and $p' \cdot x'' \geq e(p', \bar{u})$. ■

The concavity of $e(p, \bar{u})$ in p for given $\bar{u}$, which is a very important property, is actually fairly intuitive. Suppose that we initially have prices $\bar{p}$ and that $\bar{x}$ is an optimal consumption vector at these prices in the EMP. If prices change but we do not let the consumer change her consumption levels from $\bar{x}$, then the resulting expenditure will be $p \cdot \bar{x}$, which is a *linear* expression in p. But when the consumer can adjust her consumption, as in the EMP, her minimized expenditure level can be no greater than this amount. Hence, as illustrated in Figure 3.E.2(a), where we keep p_1 fixed and vary p_2, the graph of $e(p, \bar{u})$ lies below the graph of the linear function $p \cdot \bar{x}$ at all $p \neq \bar{p}$ and touches it at $\bar{p}$. This amounts to concavity because a similar relation to a linear function must hold at each point of the graph of $e(\cdot, u)$; see Figure 3.E.2(b).

Proposition 3.E.1 allows us to make an important connection between the expenditure function and the indirect utility function developed in Section 3.D. In particular, for any $p \gg 0$, $w > 0$, and $u > u(0)$ we have

$$e(p, v(p, w)) = w \quad \text{and} \quad v(p, e(p, u)) = u. \tag{3.E.1}$$

These conditions imply that for a fixed price vector $\bar{p}$, $e(\bar{p}, \cdot)$ and $v(\bar{p}, \cdot)$ are inverses to one another (see Exercise 3.E.8). In fact, in Exercise 3.E.9, you are asked to show that by using the relations in (3.E.1), Proposition 3.E.2 can be directly derived from Proposition 3.D.3, and vice versa. That is, there is a direct correspondence between the properties of the expenditure function and the indirect utility function. They both capture the same underlying features of the consumer's choice problem.

The Hicksian (or Compensated) Demand Function

The set of optimal commodity vectors in the EMP is denoted $h(p, u) \subset \mathbb{R}^L_+$ and is known as the *Hicksian*, or *compensated, demand correspondence*, or *function* if

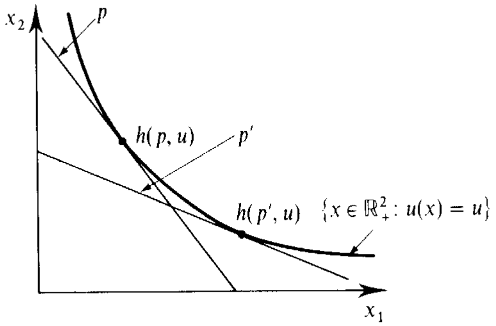


Figure 3.E.3
The Hicksian (or compensated) demand function.

single-valued. (The reason for the term "compensated demand" will be explained below.) Figure 3.E.3 depicts the solution set $h(p, u)$ for two different price vectors p and p'.

Three basic properties of Hicksian demand are given in Proposition 3.E.3, which parallels Proposition 3.D.2 for Walrasian demand.

Proposition 3.E.3: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties:

(i) *Homogeneity of degree zero in p*: $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$.
(ii) *No excess utility:* For any $x \in h(p, u)$, $u(x) = u$.
(iii) *Convexity/uniqueness:* If $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is *strictly* convex, so that $u(\cdot)$ is strictly quasiconcave, then there is a unique element in $h(p, u)$.

Proof: (i) Homogeneity of degree zero in p follows because the optimal vector when minimizing $p \cdot x$ subject to $u(x) \geq u$ is the same as that for minimizing $\alpha p \cdot x$ subject to this same constraint, for any scalar $\alpha > 0$.

(ii) This property follows from continuity of $u(\cdot)$. Suppose there exists an $x \in h(p, u)$ such that $u(x) > u$. Consider a bundle $x' = \alpha x$, where $\alpha \in (0, 1)$. By continuity, for α close enough to 1, $u(x') \geq u$ and $p \cdot x' < p \cdot x$, contradicting x being optimal in the EMP with required utility level u.

(iii) The proof of property (iii) parallels that for property (iii) of Proposition 3.D.2 and is left as an exercise (Exercise 3.E.4). ■

As in the UMP, when $u(\cdot)$ is differentiable, the optimal consumption bundle in the EMP can be characterized using first-order conditions. As would be expected given Proposition 3.E.1, these first-order conditions bear a close similarity to those of the UMP. Exercise 3.E.1 asks you to explore this relationship.

Exercise 3.E.1: Assume that $u(\cdot)$ is differentiable. Show that the first-order conditions for the EMP are

$$p \geq \lambda \nabla u(x^*) \tag{3.E.2}$$

and

$$x^* \cdot [p - \lambda \nabla u(x^*)] = 0, \tag{3.E.3}$$

for some $\lambda \geq 0$. Compare this with the first-order conditions of the UMP.

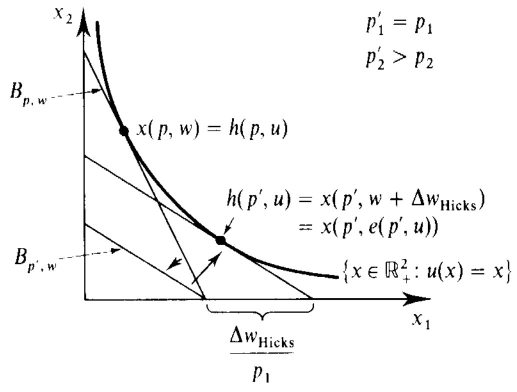


Figure 3.E.4
Hicksian wealth compensation.

We will not discuss the continuity and differentiability properties of the Hicksian demand correspondence. With minimal qualifications, they are the same as for the Walrasian demand correspondence, which we discuss in some detail in Appendix A.

Using Proposition 3.E.1, we can relate the Hicksian and Walrasian demand correspondences as follows:

$$h(p, u) = x(p, e(p, u)) \quad \text{and} \quad x(p, w) = h(p, v(p, w)). \tag{3.E.4}$$

The first of these relations explains the use of the term *compensated demand correspondence* to describe $h(p, u)$: As prices vary, $h(p, u)$ gives precisely the level of demand that would arise if the consumer's wealth were simultaneously adjusted to keep her utility level at u. This type of wealth compensation, which is depicted in Figure 3.E.4, is known as *Hicksian wealth compensation.* In Figure 3.E.4, the consumer's initial situation is the price–wealth pair (p, w), and prices then change to p', where $p'_1 = p_1$ and $p'_2 > p_2$. The Hicksian wealth compensation is the amount $\Delta w_{Hicks} = e(p', u) - w$. Thus, the demand function $h(p, u)$ keeps the consumer's utility level fixed as prices change, in contrast with the Walrasian demand function, which keeps money wealth fixed but allows utility to vary.

As with the value functions of the EMP and UMP, the relations in (3.E.4) allow us to develop a tight linkage between the properties of the Hicksian demand correspondence $h(p, u)$ and the Walrasian demand correspondence $x(p, w)$. In particular, in Exercise 3.E.10, you are asked to use the relations in (3.E.4) to derive the properties of each correspondence as a direct consequence of those of the other.

Hicksian Demand and the Compensated Law of Demand

An important property of Hicksian demand is that it satisfies the *compensated law of demand*: Demand and price move in opposite directions for price changes that are accompanied by Hicksian wealth compensation. In Proposition 3.E.4, we prove this fact for the case of single-valued Hicksian demand.

Proposition 3.E.4: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p, u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function $h(p, u)$ satisfies the compensated law of demand: For all p' and p'',

$$(p'' - p')\cdot[h(p'', u) - h(p', u)] \leq 0. \tag{3.E.5}$$

Proof: For any $p \gg 0$, consumption bundle $h(p, u)$ is optimal in the EMP, and so it achieves a lower expenditure at prices p than any other bundle that offers a utility level of at least u. Therefore, we have

$$p'' \cdot h(p'', u) \leq p'' \cdot h(p', u)$$

and

$$p' \cdot h(p'', u) \geq p' \cdot h(p', u).$$

Subtracting these two inequalities yields the results. ■

One immediate implication of Proposition 3.E.4 is that for compensated demand, own-price effects are nonpositive. In particular, if only p_ℓ changes, Proposition 3.E.4 implies that $(p''_\ell - p'_\ell)[h_\ell(p'', u) - h_\ell(p', u)] \leq 0$. The comparable statement is *not* true for Walrasian demand. Walrasian demand need not satisfy the law of demand. For example, the demand for a good can decrease when its price falls. See Section 2.E for a discussion of Giffen goods and Figure 2.F.5 (along with the discussion of that figure in Section 2.F) for a diagrammatic example.

Example 3.E.1: *Hicksian Demand and Expenditure Functions for the Cobb–Douglas Utility Function.* Suppose that the consumer has the Cobb–Douglas utility function over the two goods given in Example 3.D.1. That is, $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$. By deriving the first-order conditions for the EMP (see Exercise 3.E.1), and substituting from the constraint $u(h_1(p, u), h_2(p, u)) = u$, we obtain the Hicksian demand functions

$$h_1(p, u) = \left[\frac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha} u$$

and

$$h_2(p, u) = \left[\frac{(1-\alpha)p_1}{\alpha p_2}\right]^{\alpha} u.$$

Calculating $e(p, u) = p \cdot h(p, u)$ yields

$$e(p, u) = [\alpha^{-\alpha}(1-\alpha)^{\alpha-1}]p_1^\alpha p_2^{1-\alpha} u. \ ■$$

Exercise 3.E.2: Verify the properties listed in Propositions 3.E.2 and 3.E.3 for the Hicksian demand and expenditure functions of the Cobb–Douglas utility function.

Here and in the preceding section, we have derived several basic properties of the Walrasian and Hicksian demand functions, the indirect utility function, and the expenditure function. We investigate these concepts further in Section 3.G. First, however, in Section 3.F, which is meant as optional, we offer an introductory discussion of the mathematics underlying the theory of duality. The material covered in Section 3.F provides a better understanding of the essential connections between the UMP and the EMP. We emphasize, however, that this section is not a prerequisite for the study of the remaining sections of this chapter.

3.F Duality: A Mathematical Introduction

This section constitutes a mathematical detour. It focuses on some aspects of the theory of convex sets and functions.

Recall that a set $K \subset \mathbb{R}^L$ is convex if $\alpha x + (1-\alpha)z \in K$ whenever $x, z \in K$ and $\alpha \in [0, 1]$. Note that the intersection of two convex sets is a convex set.

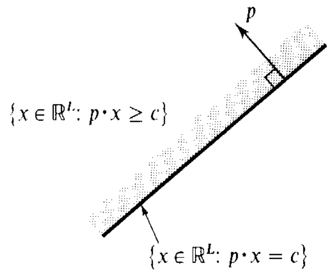


Figure 3.F.1
A half-space and a hyperplane.

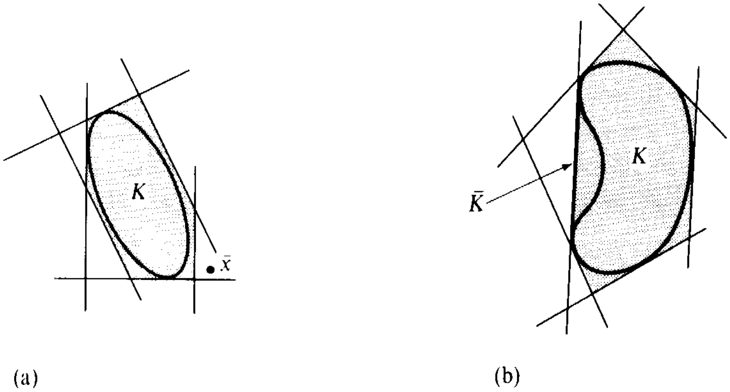


Figure 3.F.2
A closed set is convex if and only if it equals the intersection of the half-spaces that contain it.
(a) Convex K.
(b) Nonconvex K.

A *half-space* is a set of the form $\{x \in \mathbb{R}^L: p \cdot x \geq c\}$ for some $p \in \mathbb{R}^L$, $p \neq 0$, called the *normal vector* to the half-space, and some $c \in \mathbb{R}$. Its boundary $\{x \in \mathbb{R}^L: p \cdot x = c\}$ is called a *hyperplane*. The term *normal* comes from the fact that whenever $p \cdot x = p \cdot x' = c$, we have $p \cdot (x - x') = 0$, and so p is orthogonal (i.e., perpendicular, or normal) to the hyperplane (see Figure 3.F.1). Note that both half-spaces and hyperplanes are convex sets.

Suppose now that $K \subset \mathbb{R}^L$ is a convex set that is also closed (i.e., it includes its boundary points), and consider any point $\bar{x} \notin K$ outside of this set. A fundamental theorem of convexity theory, the *separating hyperplane theorem*, tells us that there is a half-space containing K and excluding $\bar{x}$ (see Section M.G of the Mathematical Appendix). That is, there is a $p \in \mathbb{R}^L$ and a $c \in \mathbb{R}$ such that $p \cdot \bar{x} < c \leq p \cdot x$ for all $x \in K$. The basic idea behind duality theory is the fact that a closed, convex set can equivalently ("dually") be described as the intersection of the half-spaces that contain it; this is illustrated in Figure 3.F.2(a). Because any $\bar{x} \notin K$ is excluded by some half-space that contains K, as we draw such half-spaces for more and more points $\bar{x} \notin K$, their intersection (the shaded area in the figure) becomes equal to K.

More generally, if the set K is not convex, the intersection of the half-spaces that contain K is the smallest closed, convex set that contains K, known as the *closed, convex hull* of K. Figure 3.F.2(b) illustrates a case where the set K is nonconvex; in the figure, the closed convex hull of K is $\bar{K}$.

Given any closed (but not necessarily convex) set $K \subset \mathbb{R}^L$ and a vector $p \in \mathbb{R}^L$, we can define the *support function* of K.

Definition 3.F.1: For any nonempty closed set $K \subset \mathbb{R}^L$, the *support function* of K is defined for any $p \in \mathbb{R}^L$ to be

$$\mu_K(p) = \text{Infimum}\,\{p \cdot x: x \in K\}.$$

The *infimum* of a set of numbers, as used in Definition 3.F.1, is a generalized version of the set's minimum value. In particular, it allows for situations in which no minimum exists because although points in the set can be found that come arbitrarily close to some lower bound value, no point in the set actually attains that value. For example, consider a strictly positive function $f(x)$ that approaches zero asymptotically as x increases. The minimum of this function does not exist, but its infimum is zero. The formulation also allows $\mu_K(p)$ to take the value $-\infty$ when points in K can be found that make the value of $p \cdot x$ unboundedly negative.

When K is convex, the function $\mu_K(\cdot)$ provides an alternative ("dual") description of K because we can reconstruct K from knowledge of $\mu_K(\cdot)$. In particular, for every p, $\{x \in \mathbb{R}^L: p \cdot x \geq \mu_K(p)\}$ is a half-space that contains K. In addition, as we discussed above, if $x \notin K$, then $p \cdot x < \mu_K(p)$ for some p. Thus, the intersection of the half-spaces generated by all possible values of p is precisely K; that is,

$$K = \{x \in \mathbb{R}^L: p \cdot x \geq \mu_K(p) \text{ for every } p\}.$$

By the same logic, if K is not convex, then $\{x \in \mathbb{R}^L: p \cdot x \geq \mu_K(p) \text{ for every } p\}$ is the smallest closed, convex set containing K.

The function $\mu_K(\cdot)$ is homogeneous of degree one. More interestingly, *it is concave.* To see this, consider $p'' = \alpha p + (1 - \alpha)p'$ for $\alpha \in [0,1]$. To make things simple, suppose that the infimum is in fact attained, so that there is a $z \in K$ such that $\mu_K(p'') = p'' \cdot z$. Then, because

$$\begin{aligned}\mu_K(p'') &= \alpha p \cdot z + (1 - \alpha)p' \cdot z \\ &\geq \alpha \mu_K(p) + (1 - \alpha)\mu_K(p').\end{aligned}$$

we conclude that $\mu_K(\cdot)$ is concave.

The concavity of $\mu_K(\cdot)$ can also be seen geometrically. Figure 3.F.3 depicts the value of the function $\phi_x(p) = p \cdot x$, for various choices of $x \in K$, as a function of p_2 (with p_1 fixed at $\bar{p}_1$). For each x, the function $\phi_x(\cdot)$ is a linear function of p_2. Also shown in the figure is $\mu_K(\cdot)$. For each level of p_2, $\mu_K(\bar{p}_1, p_2)$ is equal to the minimum value (technically, the infimum) of the various linear functions $\phi_x(\cdot)$ at $p = (\bar{p}_1, p_2)$; that is, $\mu_K(\bar{p}_1, p_2) = \text{Min}\{\phi_x(\bar{p}_1, p_2): x \in K\}$. For example, when $p_2 = \bar{p}_2$, $\mu_K(\bar{p}_1, \bar{p}_2) = \phi_{\bar{x}}(\bar{p}_1, \bar{p}_2) \leq \phi_x(\bar{p}_1, \bar{p}_2)$ for all $x \in K$. As can be seen in the figure, $\mu_K(\cdot)$ is therefore the "lower envelope" of the functions $\phi_x(\cdot)$. As the infimum of a family of linear functions, $\mu_K(\cdot)$ is concave.

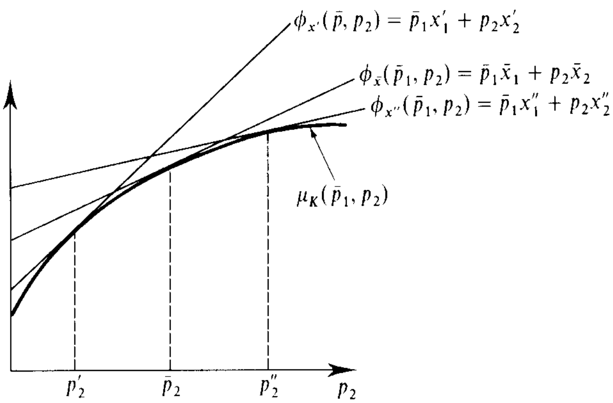


Figure 3.F.3
The support function $\mu_K(p)$ is concave.

Proposition 3.F.1, the *duality theorem*, gives the central result of the mathematical theory. Its use is pervasive in economics.

Proposition 3.F.1: (*The Duality Theorem*). Let K be a nonempty closed set, and let $\mu_K(\cdot)$ be its support function. Then there is a unique $\bar{x} \in K$ such that $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$ if and only if $\mu_K(\cdot)$ is differentiable at $\bar{p}$. Moreover, in this case,

$$\nabla \mu_K(\bar{p}) = \bar{x}.$$

We will not give a complete proof of the theorem. Its most important conclusion is that if the minimizing vector $\bar{x}$ for the vector $\bar{p}$ is unique, then the gradient of the support function at $\bar{p}$ is equal to $\bar{x}$. To understand this result, consider the linear function $\phi_{\bar{x}}(p) = p \cdot \bar{x}$. By the definition of $\bar{x}$, we know that $\mu_K(\bar{p}) = \phi_{\bar{x}}(\bar{p})$. Moreover, the derivatives of $\phi_{\bar{x}}(\cdot)$ at $\bar{p}$ satisfy $\nabla \phi_{\bar{x}}(p) = \bar{x}$. Therefore, the duality theorem tells us that as far as the first derivatives of $\mu_K(\cdot)$ are concerned, it is as if $\mu_K(\cdot)$ is linear in p; that is, the first derivatives of $\mu_K(\cdot)$ at $\bar{p}$ are exactly the same as those of the function $\phi_{\bar{x}}(p) = p \cdot \bar{x}$.

The logic behind this fact is relatively straightforward. Suppose that $\mu_K(\cdot)$ is differentiable at $\bar{p}$, and consider the function $\xi(p) = p \cdot \bar{x} - \mu_K(p)$, where $\bar{x} \in K$ and $\mu_K(\bar{p}) = \bar{p} \cdot \bar{x}$. By the definition of $\mu_K(\cdot)$, $\xi(p) = p \cdot \bar{x} - \mu_K(p) \geq 0$ for all p. We also know that $\xi(\bar{p}) = \bar{p} \cdot \bar{x} - \mu_K(\bar{p}) = 0$. So the function $\xi(\cdot)$ reaches a minimum at $p = \bar{p}$. As a result, its partial derivatives at $\bar{p}$ must all be zero. This implies the result: $\nabla \xi(\bar{p}) = \bar{x} - \nabla \mu_K(\bar{p}) = 0$.[14]

Recalling our discussion of the EMP in Section 3.E, we see that the expenditure function is precisely the support function of the set $\{x \in \mathbb{R}^L_+ : u(x) \geq u\}$. From our discussion of the support function, several of the properties of the expenditure function previously derived in Proposition 3.E.2, such as homogeneity of degree zero and concavity, immediately follow. In Section 3.G, we study the implications of the duality theorem for the theory of demand.

For a further discussion of duality theory and its applications, see Green and Heller (1981) and, for an advanced treatment, Diewert (1982). For an early application of duality to consumer theory, see McKenzie (1956–57).

The first part of the duality theorem says that $\mu_K(\cdot)$ is differentiable at $\bar{p}$ if and only if the minimizing vector at $\bar{p}$ is unique. If K is not strictly convex, then at some $\bar{p}$, the minimizing vector will not be unique and therefore $\mu_K(\cdot)$ will exhibit a kink at $\bar{p}$. Nevertheless, in a sense that can be made precise by means of the concept of directional derivatives, the gradient $\mu_K(\cdot)$ at this p is still equal to the minimizing set, which in this case is multivalued.

This is illustrated in Figure 3.F.4 for $L = 2$. In panel (a) of Figure 3.F.4, a strictly convex set K is depicted. For all p, its minimizing vector is unique. At $\bar{p} = (\frac{1}{2}, \frac{1}{2})$, it is $\bar{x} = (1, 1)$. Panel (b) of Figure 3.F.4 graphs $\mu_K(\frac{1}{2}, p_2)$ as a function of p_2. As can be seen, the function is concave and differentiable in p_2, with a slope of 1 (the value of $\bar{x}_2$) at $p_2 = \frac{1}{2}$.

In panel (a) of Figure 3.F.5, a convex but not strictly convex set K is depicted. At $p = (\frac{1}{2}, \frac{1}{2})$, the entire segment $[x', x'']$ is the minimizing set. If $p_1 > p_2$, then x' is the minimizing vector and the value of the support function is $p_1 x'_1 + p_2 x'_2$, whereas if $p_1 < p_2$, then x'' is optimal and the value of the support function is $p_1 x''_1 + p_2 x''_2$. Panel (b) of Figure 3.F.5

14. Because $\bar{x} = \nabla \mu_K(\bar{p})$ for any minimizer $\bar{x}$ at $\bar{p}$, either $\bar{x}$ is unique or if it is not unique, then $\mu_K(\cdot)$ could not be differentiable at $\bar{p}$. Thus, $\mu_K(\cdot)$ is differentiable at $\bar{p}$ only if there is a unique minimizer at $\bar{p}$.

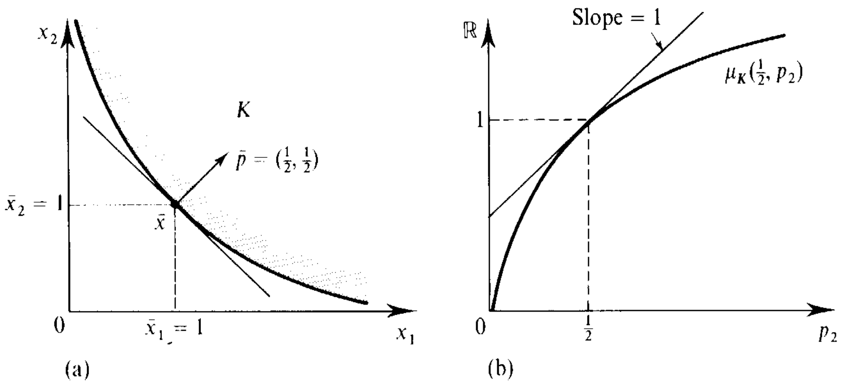


Figure 3.F.4
The duality theorem with a unique minimizing vector at $\bar{p}$. (a) The minimum vector. (b) The support function.

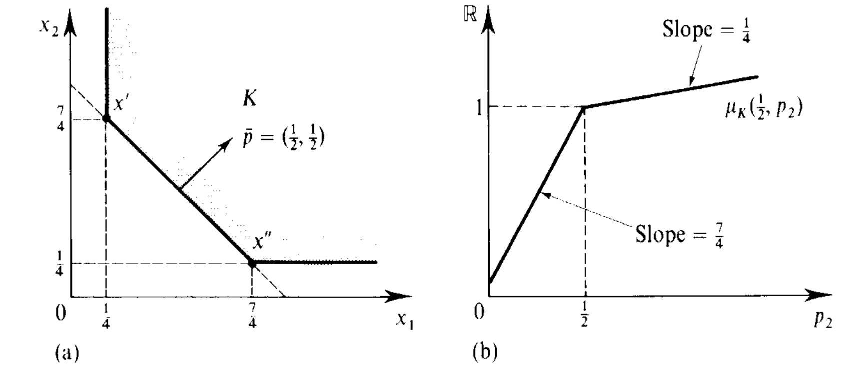


Figure 3.F.5
The duality theorem with a multivalued minimizing set at $\bar{p}$. (a) The minimum set. (b) The support function.

graphs $\mu_K(\frac{1}{2}, p_2)$ as a function of p_2. For $p_2 < \frac{1}{2}$, its slope is equal to $\frac{7}{4}$, the value of x'_2. For $p_2 > \frac{1}{2}$, its slope is $\frac{1}{4}$, the value of x''_2. There is a kink in the function at $\bar{p} = (\frac{1}{2}, \frac{1}{2})$, the price vector that has multiple minimizing vectors, with its left derivative with respect to p_2 equal to $\frac{7}{4}$ and its right derivative equal to $\frac{1}{4}$. Thus, the range of these directional derivatives at $\bar{p} = (\frac{1}{2}, \frac{1}{2})$ is equal to the range of x_2 in the minimizing vectors at that point.

3.G Relationships between Demand, Indirect Utility, and Expenditure Functions

We now continue our exploration of results flowing from the UMP and the EMP. The investigation in this section concerns three relationships: that between the Hicksian demand function and the expenditure function, that between the Hicksian and Walrasian demand functions, and that between the Walrasian demand function and the indirect utility function.

As before, we assume that $u(\cdot)$ is a continuous utility function representing the locally nonsatiated preferences $\succsim$ (defined on the consumption set $X = \mathbb{R}^L_+$), and we restrict attention to cases where $p \gg 0$. In addition, to keep matters simple, we assume throughout that $\succsim$ is strictly convex, so that the Walrasian and Hicksian demands, $x(p, w)$ and $h(p, u)$, are single-valued.[15]

Hicksian Demand and the Expenditure Function

From knowledge of the Hicksian demand function, the expenditure function can readily be calculated as $e(p, u) = p \cdot h(p, u)$. The important result shown in Proposition 3.G.1 establishes a more significant link between the two concepts that runs in the opposite direction.

Proposition 3.G.1: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. For all p and u, the Hicksian demand $h(p, u)$ is the derivative vector of the expenditure function with respect to prices:

$$h(p, u) = \nabla_p e(p, u). \tag{3.G.1}$$

That is, $h_\ell(p, u) = \partial e(p, u)/\partial p_\ell$ for all $\ell = 1, \dots, L$.

Thus, given the expenditure function, we can calculate the consumer's Hicksian demand function simply by differentiating.

We provide three proofs of this important result.

Proof 1: (*Duality Theorem Argument*). The result is an immediate consequence of the duality theorem (Proposition 3.F.1). Since the expenditure function is precisely the support function for the set $K = \{x \in \mathbb{R}^L_+ : u(x) \geq u\}$, and since the optimizing vector associated with this support function is $h(p, u)$, Proposition 3.F.1 implies that $h(p, u) = \nabla_p e(p, u)$. Note that (3.G.1) helps us understand the use of the term "dual" in this context. In particular, just as the derivatives of the utility function $u(\cdot)$ with respect to quantities have a price interpretation (we have seen in Section 3.D that at an optimum they are equal to prices multiplied by a constant factor of proportionality), (3.G.1) tells us that the derivatives of the expenditure function $e(\cdot, u)$ with respect to prices have a quantity interpretation (they are equal to the Hicksian demands). ■

Proof 2: (*First-Order Conditions Argument*). For this argument, we focus for simplicity on the case where $h(p, u) \gg 0$, and we assume that $h(p, u)$ is differentiable at (p, u).

Using the chain rule, the change in expenditure can be written as

$$\begin{aligned} \nabla_p e(p, u) &= \nabla_p [p \cdot h(p, u)] \\ &= h(p, u) + [p \cdot D_p h(p, u)]^{\mathrm{T}}. \end{aligned} \tag{3.G.2}$$

Substituting from the first-order conditions for an interior solution to the EMP, $p = \lambda \nabla u(h(p, u))$, yields

$$\nabla_p e(p, u) = h(p, u) + \lambda [\nabla u(h(p, u)) \cdot D_p h(p, u)]^{\mathrm{T}}.$$

But since the constraint $u(h(p, u)) = u$ holds for all p in the EMP, we know that $\nabla u(h(p, u)) \cdot D_p h(p, u) = 0$, and so we have the result. ■

15. In fact, all the results of this section are local results that hold at all price vectors $\bar{p}$ with the property that for all p near $\bar{p}$, the optimal consumption vector in the UMP or EMP with price vector p is unique.

Proof 3: (*Envelope Theorem Argument*). Under the same simplifying assumptions used in Proof 2, we can directly appeal to the *envelope theorem*. Consider the value function $\phi(\alpha)$ of the constrained minimization problem

$$\begin{aligned} \underset{x}{\text{Min}} \quad & f(x, \alpha) \\ \text{s.t. } & g(x, \alpha) = 0. \end{aligned}$$

If $x^*(\alpha)$ is the (differentiable) solution to this problem as a function of the parameters $\alpha = (\alpha_1, \ldots, \alpha_M)$, then the envelope theorem tells us that at any $\bar{\alpha} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_M)$ we have

$$\frac{\partial \phi(\bar{\alpha})}{\partial \alpha_m} = \frac{\partial f(x^*(\bar{\alpha}), \bar{\alpha})}{\partial \alpha_m} - \lambda \frac{\partial g(x^*(\bar{\alpha}), \bar{\alpha})}{\partial \alpha_m}$$

for $m = 1, \ldots, M$, or in matrix notation,

$$\nabla_\alpha \phi(\bar{\alpha}) = \nabla_\alpha f(x^*(\bar{\alpha}), \bar{\alpha}) - \lambda \nabla_\alpha g(x^*(\bar{\alpha}), \bar{\alpha}).$$

See Section M.L of the Mathematical Appendix for a further discussion of this result.[16]

Because prices are parameters in the EMP that enter only the objective function $p \cdot x$, the change in the value function of the EMP with respect to a price change at $\bar{p}$, $\nabla_p e(\bar{p}, u)$, is just the vector of partial derivatives with respect to p of the objective function evaluated at the optimizing vector, $h(\bar{p}, u)$. Hence $\nabla_p e(p, u) = h(p, u)$. ■

The idea behind all three proofs is the same: If we are at an optimum in the EMP, the changes in demand caused by price changes have no first-order effect on the consumer's expenditure. This can be most clearly seen in Proof 2; condition (3.G.2) uses the chain rule to break the total effect of the price change into two effects: a direct effect on expenditure from the change in prices holding demand fixed (the first term) and an indirect effect on expenditure caused by the induced change in demand holding prices fixed (the second term). However, because we are at an expenditure minimizing bundle, the first-order conditions for the EMP imply that this latter effect is zero.

Proposition 3.G.2 summarizes several properties of the price derivatives of the Hicksian demand function $D_p h(p, u)$ that are implied by Proposition 3.G.1 [properties (i) to (iii)]. It also records one additional fact about these derivatives [property (iv)].

Proposition 3.G.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that $h(\cdot, u)$ is continuously differentiable at (p, u), and denote its $L \times L$ derivative matrix by $D_p h(p, u)$. Then

(i) $D_p h(p, u) = D_p^2 e(p, u)$.
(ii) $D_p h(p, u)$ is a negative semidefinite matrix.
(iii) $D_p h(p, u)$ is a symmetric matrix.
(iv) $D_p h(p, u) p = 0$.

Proof: Property (i) follows immediately from Proposition 3.G.1 by differentiation. Properties (ii) and (iii) follow from property (i) and the fact that since $e(p, u)$ is a

16. Proof 2 is essentially a proof of the envelope theorem for the special case where the parameters being changed (in this case, prices) affect only the objective function of the problem.

twice continuously differentiable concave function, it has a symmetric and negative semidefinite Hessian (i.e., second derivative) matrix (see Section M.C of the Mathematical Appendix). Finally, for property (iv), note that because $h(p, u)$ is homogeneous of degree zero in p, $h(\alpha p, u) - h(p, u) = 0$ for all α; differentiating this expression with respect to α yields $D_p h(p, u)p = 0$. [Note that because $h(p, u)$ is homogeneous of degree zero, $D_p h(p, u)p = 0$ also follows directly from Euler's formula; see Section M.B of the Mathematical Appendix.] ■

The negative semidefiniteness of $D_p h(p, u)$ is the differential analog of the compensated law of demand, condition (3.E.5). In particular, the differential version of (3.E.5) is $dp \cdot dh(p, u) \leq 0$. Since $dh(p, u) = D_p h(p, u)\, dp$, substituting gives $dp \cdot D_p h(p, u)\, dp \leq 0$ for all dp; therefore, $D_p h(p, u)$ is negative semidefinite. Note that negative semidefiniteness implies that $\partial h_\ell(p, u)/\partial p_\ell \leq 0$ for all ℓ; that is, compensated own-price effects are nonpositive, a conclusion that we have also derived directly from condition (3.E.5).

The symmetry of $D_p h(p, u)$ is an unexpected property. It implies that compensated price cross-derivatives between any two goods ℓ and k must satisfy $\partial h_\ell(p, u)/\partial p_k = \partial h_k(p, u)/\partial p_\ell$. Symmetry is not easy to interpret in plain economic terms. As emphasized by Samuelson (1947), it is a property just beyond what one would derive without the help of mathematics. Once we know that $D_p h(p, u) = \nabla_p^2 e(p, u)$, the symmetry property reflects the fact that the cross derivatives of a (twice continuously differentiable) function are equal. In intuitive terms, this says that when you climb a mountain, you will cover the same net height regardless of the route.[17] As we discuss in Sections 13.H and 13.J, this path-independence feature is closely linked to the transitivity, or "no-cycling", aspect of rational preferences.

We define two goods ℓ and k to be *substitutes* at (p, u) if $\partial h_\ell(p, u)/\partial p_k \geq 0$ and *complements* if this derivative is nonpositive [when Walrasian demands have these relationships at (p, w), the goods are referred to as *gross* substitutes and *gross* complements at (p, w), respectively]. Because $\partial h_\ell(p, u)/\partial p_\ell \leq 0$, property (iv) of Proposition 3.G.2 implies that there must be a good k for which $\partial h_\ell(p, u)/\partial p_k \geq 0$. Hence, Proposition 3.G.2 implies that every good has at least one substitute.

17. To see why this is so, consider the twice continuously differentiable function $f(x, y)$. We can express the change in this function's value from (x', y') to (x'', y'') as the summation (technically, the integral) of two different paths of incremental change: $f(x'', y'') - f(x', y') = \int_{y'}^{y''} [\partial f(x', t)/\partial y]\, dt + \int_{x'}^{x''} [\partial f(s, y'')/\partial x]\, ds$ and $f(x'', y'') - f(x', y') = \int_{x'}^{x''} [\partial f(s, y')/\partial x]\, ds + \int_{y'}^{y''} [\partial f(x'', t)/\partial y]\, dt$. For these two to be equal (as they must be), we should have

$$\int_{y'}^{y''} \left[\frac{\partial f(x'', t)}{\partial y} - \frac{\partial f(x'', t)}{\partial y}\right] dt = \int_{x'}^{x''} \left[\frac{\partial f(s, y'')}{\partial x} - \frac{\partial f(s, y')}{\partial x}\right] ds$$

or

$$\int_{y'}^{y''} \left\{\int_{x'}^{x''} \left[\frac{\partial^2 f(s, t)}{\partial y\, \partial x}\right] ds\right\} dt = \int_{x'}^{x''} \left\{\int_{y'}^{y''} \left[\frac{\partial^2 f(s, t)}{\partial x\, \partial y}\right] dt\right\} ds.$$

So equality of cross-derivatives implies that these two different ways of "climbing the function" yield the same result. Likewise, if the cross-partials were not equal to (x'', y''), then for (x', y') close enough to (x'', y''), the last equality would be violated.

The Hicksian and Walrasian Demand Functions

Although the Hicksian demand function is not directly observable (it has the consumer's utility level as an argument), we now show that $D_p h(p, u)$ can nevertheless be computed from the observable Walrasian demand function $x(p, w)$ (its arguments are all observable in principle). This important result, known as the *Slutsky equation*, means that the properties listed in Proposition 3.G.2 translate into restrictions on the observable Walrasian demand function $x(p, w)$.

Proposition 3.G.3: (*The Slutsky Equation*) Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for all (p, w), and $u = v(p, w)$, we have

$$\frac{\partial h_\ell(p, u)}{\partial p_k} = \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w) \quad \text{for all } \ell, k \qquad (3.G.3)$$

or equivalently, in matrix notation,

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w) x(p, w)^{\mathrm{T}}. \qquad (3.G.4)$$

Proof: Consider a consumer facing the price–wealth pair $(\bar{p}, \bar{w})$ and attaining utility level $\bar{u}$. Note that her wealth level $\bar{w}$ must satisfy $\bar{w} = e(\bar{p}, \bar{u})$. From condition (3.E.4), we know that for all (p, u), $h_\ell(p, u) = x_\ell(p, e(p, u))$. Differentiating this expression with respect to p_k and evaluating it at $(\bar{p}, \bar{u})$, we get

$$\frac{\partial h_\ell(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_\ell(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \frac{\partial x_\ell(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}.$$

Using Proposition 3.G.1, this yields

$$\frac{\partial h_\ell(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_\ell(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \frac{\partial x_\ell(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} h_k(\bar{p}, \bar{u}).$$

Finally, since $\bar{w} = e(\bar{p}, \bar{u})$ and $h_k(\bar{p}, \bar{u}) = x_k(\bar{p}, e(\bar{p}, \bar{u})) = x_k(\bar{p}, \bar{w})$, we have

$$\frac{\partial h_\ell(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_\ell(\bar{p}, \bar{w})}{\partial p_k} + \frac{\partial x_\ell(\bar{p}, \bar{w})}{\partial w} x_k(\bar{p}, \bar{w}). \quad \blacksquare$$

Figure 3.G.1(a) depicts the Walrasian and Hicksian demand curves for good ℓ as a function of p_ℓ, holding other prices fixed at $\bar{p}_{-\ell}$ [we use $\bar{p}_{-\ell}$ to denote a vector

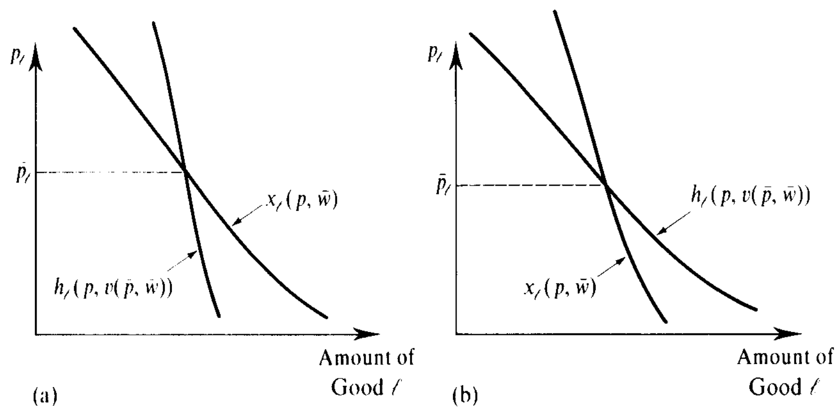


Figure 3.G.1
The Walrasian and Hicksian demand functions for good ℓ.
(a) Normal good.
(b) Inferior good.

including all prices other than p_ℓ and abuse notation by writing the price vector as $p = (p_\ell, \bar{p}_{-\ell})$]. The figure shows the Walrasian demand function $x(p, \bar{w})$ and the Hicksian demand function $h(p, \bar{u})$ with required utility level $\bar{u} = v((\bar{p}_\ell, \bar{p}_{-\ell}), \bar{w})$. Note that the two demand functions are equal when $p_\ell = \bar{p}_\ell$. The Slutsky equation describes the relationship between the slopes of these two functions at price $\bar{p}_\ell$. In Figure 3.G.1(a), the slope of the Walrasian demand curve at $\bar{p}_\ell$ is less negative than the slope of the Hicksian demand curve at that price. From inspection of the Slutsky equation, this corresponds to a situation where good ℓ is a normal good at $(\bar{p}, \bar{w})$. When p_ℓ increases above $\bar{p}_\ell$, we must increase the consumer's wealth if we are to keep her at the same level of utility. Therefore, if good ℓ is normal, its demand falls by more in the absence of this compensation. Figure 3.G.1(b) illustrates a case in which good ℓ is an inferior good. In this case, the Walrasian demand curve has a more negative slope than the Hicksian curve.

Proposition 3.G.3 implies that the matrix of price derivatives $D_p h(p, u)$ of the Hicksian demand function is equal to the matrix

$$S(p, w) = \begin{bmatrix} s_{11}(p, w) & \cdots & s_{1L}(p, w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p, w) & \cdots & s_{LL}(p, w) \end{bmatrix},$$

with $s_{\ell k}(p, w) = \partial x_\ell(p, w)/\partial p_k + [\partial x_\ell(p, w)/\partial w] x_k(p, w)$. This matrix is known as the *Slutsky substitution matrix*. Note, in particular, that $S(p, w)$ is directly computable from knowledge of the (observable) Walrasian demand function $x(p, w)$. Because $S(p, w) = D_p h(p, u)$, Proposition 3.G.2 implies that when demand is generated from preference maximization, $S(p, w)$ must possess the following three properties: it must be *negative semidefinite, symmetric, and satisfy $S(p, w)p = 0$.*

In Section 2.F, the Slutsky substitution matrix $S(p, w)$ was shown to be the matrix of compensated demand derivatives arising from a different form of wealth compensation, the so-called *Slutsky wealth compensation*. Instead of varying wealth to keep utility fixed, as we do here, Slutsky compensation adjusts wealth so that the initial consumption bundle $\bar{x}$ is just affordable at the new prices. Thus, we have the remarkable conclusion that the *derivative of the Hicksian demand function is equal to the derivative of this alternative Slutsky compensated demand.*

We can understand this result as follows: Suppose we have a utility function $u(\cdot)$ and are at initial position $(\bar{p}, \bar{w})$ with $\bar{x} = x(\bar{p}, \bar{w})$ and $\bar{u} = u(\bar{x})$. As we change prices to p', we want to change wealth in order to compensate for the wealth effect arising from this price change. In principle, the compensation can be done in two ways. By changing wealth by amount $\Delta w_{\text{Slutsky}} = p' \cdot x(\bar{p}, \bar{w}) - \bar{w}$, we leave the consumer just able to afford her initial bundle $\bar{x}$. Alternatively, we can change wealth by amount $\Delta w_{\text{Hicks}} = e(p', \bar{u}) - \bar{w}$ to keep her utility level unchanged. We have $\Delta w_{\text{Hicks}} \leq \Delta w_{\text{Slutsky}}$, and the inequality will, in general, be strict for any discrete change (see Figure 3.G.2). But because $\nabla_p e(\bar{p}, \bar{u}) = h(\bar{p}, \bar{u}) = x(\bar{p}, \bar{w})$, these two compensations are *identical* for a differential price change starting at $\bar{p}$. Intuitively, this is due to the same fact that led to Proposition 3.G.1: For a differential change in prices, the total effect on the expenditure required to achieve utility level $\bar{u}$ (the Hicksian compensation level) is simply the direct effect of the price change, assuming that the consumption bundle $\bar{x}$ does not change. But this is precisely the calculation done for Slutsky compensation. Hence, the derivatives of the compensated demand functions that arise from these two compensation mechanisms are the same.

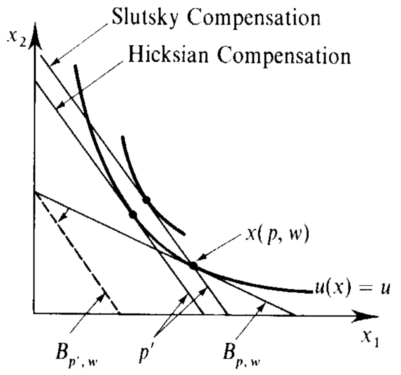


Figure 3.G.2 Hicksian versus Slutsky wealth compensation.

The fact that $D_p h(p, u) = S(p, w)$ allows us to compare the implications of the preference-based approach to consumer demand with those derived in Section 2.F using a choice-based approach built on the weak axiom. Our discussion in Section 2.F concluded that if $x(p, w)$ satisfies the weak axiom (plus homogeneity of degree zero and Walras' law), then $S(p, w)$ is negative semidefinite with $S(p, w)p = 0$. Moreover, we argued that except when $L = 2$, demand satisfying the weak axiom need not have a symmetric Slutsky substitution matrix. Therefore, the results here tell us that the restrictions imposed on demand in the preference-based approach are stronger than those arising in the choice-based theory built on the weak axiom. In fact, it is impossible to find preferences that rationalize demand when the substitution matrix is not symmetric. In Section 3.I, we explore further the role that this symmetry property plays in the relation between the preference and choice-based approaches to demand.

Walrasian Demand and the Indirect Utility Function

We have seen that the minimizing vector of the EMP, $h(p, u)$, is the derivative with respect to p of the EMP's value function $e(p, u)$. The exactly analogous statement for the UMP does not hold. The Walrasian demand, an ordinal concept, cannot equal the price derivative of the indirect utility function, which is not invariant to increasing transformations of utility. But with a small correction in which we normalize the derivatives of $v(p, w)$ with respect to p by the marginal utility of wealth, it holds true. This proposition, called *Roy's identity* (after René Roy), is the parallel result to Proposition 3.G.1 for the demand and value functions of the UMP. As with Proposition 3.G.1, we offer several proofs.

Proposition 3.G.4: (*Roy's Identity*). Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\bar{p}, \bar{w}) \gg 0$. Then

$$x(\bar{p}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w}).$$

That is, for every $\ell = 1, \ldots, L$:

$$x_\ell(\bar{p}, \bar{w}) = -\frac{\partial v(\bar{p}, \bar{w})/\partial p_\ell}{\partial v(\bar{p}, \bar{w})/\partial w}.$$

Proof 1: Let $\bar{u} = v(\bar{p}, \bar{w})$. Because the identity $v(p, e(p, \bar{u})) = \bar{u}$ holds for all p, differentiating with respect to p and evaluating at $p = \bar{p}$ yields

$$\nabla_p v(\bar{p}, e(\bar{p}, \bar{u})) + \frac{\partial v(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \nabla_p e(\bar{p}, \bar{u}) = 0.$$

But $\nabla_p e(\bar{p}, \bar{u}) = h(\bar{p}, \bar{u})$ by Proposition 3.G.1, and and so we can substitute and get

$$\nabla_p v(\bar{p}, e(\bar{p}, \bar{u})) + \frac{\partial v(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} h(\bar{p}, \bar{u}) = 0.$$

Finally, since $\bar{w} = e(\bar{p}, \bar{u})$, we can write

$$\nabla_p v(\bar{p}, \bar{w}) + \frac{\partial v(\bar{p}, \bar{w})}{\partial w} x(\bar{p}, \bar{w}) = 0.$$

Rearranging, this yields the result. ■

Proof 1 of Roy's identity derives the result using Proposition 3.G.1. Proofs 2 and 3 highlight the fact that both results actually follow from the same idea: Because we are at an optimum, the demand response to a price change can be ignored in calculating the effect of a differential price change on the value function. Thus, Roy's identity and Proposition 3.G.1 should be viewed as parallel results for the UMP and EMP. (Indeed, Exercise 3.G.1 asks you to derive Proposition 3.G.1 as a consequence of Roy's identity, thereby showing that the direction of the argument in Proof 1 can be reversed.)

Proof 2: (*First-Order Conditions Argument*). Assume that $x(p, w)$ is differentiable and $x(\bar{p}, \bar{w}) \gg 0$. By the chain rule, we can write

$$\frac{\partial v(\bar{p}, \bar{w})}{\partial p_\ell} = \sum_{k=1}^{L} \frac{\partial u(x(\bar{p}, \bar{w}))}{\partial x_k} \frac{\partial x_k(\bar{p}, \bar{w})}{\partial p_\ell}.$$

Substituting for $\partial u(x(\bar{p}, \bar{w}))/\partial x_k$ using the first-order conditions for the UMP, we have

$$\begin{aligned}\frac{\partial v(\bar{p}, \bar{w})}{\partial p_\ell} &= \sum_{k=1}^{L} \lambda p_k \frac{\partial x_k(\bar{p}, \bar{w})}{\partial p_\ell} \\ &= -\lambda x_\ell(\bar{p}, \bar{w}),\end{aligned}$$

since $\sum_k p_k(\partial x_k(\bar{p}, \bar{w})/\partial p_\ell) = -x_\ell(\bar{p}, \bar{w})$ (Proposition 2.E.2). Finally, we have already argued that $\lambda = \partial v(\bar{p}, \bar{w})/\partial w$ (see Section 3.D); use of this fact yields the result. ■

Proof 2 is again essentially a proof of the envelope theorem, this time for the case where the parameter that varies enters only the constraint. The next result uses the envelope theorem directly.

Proof 3: (*Envelope Theorem Argument*) Applied to the UMP, the envelope theorem tells us directly that the utility effect of a marginal change in p_ℓ is equal to its effect on the consumer's budget constraint weighted by the Lagrange multiplier λ of the consumer's wealth constraint. That is, $\partial v(\bar{p}, \bar{w})/\partial p_\ell = -\lambda x_\ell(\bar{p}, \bar{w})$. Similarly, the utility effect of a differential change in wealth $\partial v(p, w)/\partial w$ is just λ. Combining these two facts yields the result. ■

Proposition 3.G.4 provides a substantial payoff. Walrasian demand is much easier to compute from indirect than from direct utility. To derive $x(p, w)$ from the indirect

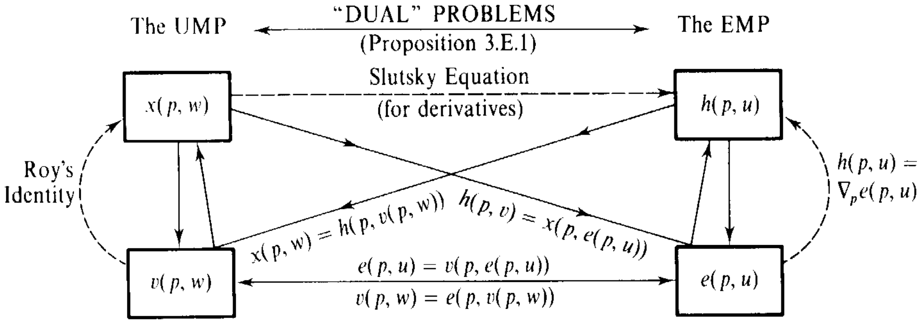


Figure 3.G.3 Relationships between the UMP and the EMP.

utility function, no more than the calculation of derivatives is involved; no system of first-order condition equations needs to be solved. Thus, it may often be more convenient to express tastes in indirect utility form. In Chapter 4, for example, we will be interested in preferences with the property that wealth expansion paths are linear over some range of wealth. It is simple to verify using Roy's identity that indirect utilities of the *Gorman* form $v(p, w) = a(p) + b(p)w$ have this property (see Exercise 3.G.11).

Figure 3.G.3 summarizes the connection between the demand and value functions arising from the UMP and the EMP; a similar figure appears in Deaton and Muellbauer (1980). The solid arrows indicate the derivations discussed in Sections 3.D and 3.E. Starting from a given utility function in the UMP or the EMP, we can derive the optimal consumption bundles $x(p, w)$ and $h(p, u)$ and the value functions $v(p, w)$ and $e(p, u)$. In addition, we can go back and forth between the value functions and demand functions of the two problems using relationships (3.E.1) and (3.E.4).

The relationships developed in this section are represented in Figure 3.G.3 by dashed arrows. We have seen here that the demand vector for each problem can be calculated from its value function and that the derivatives of the Hicksian demand function can be calculated from the observable Walrasian demand using Slutsky's equation.

3.H Integrability

If a continuously differentiable demand function $x(p, w)$ is generated by rational preferences, then we have seen that it must be homogeneous of degree zero, satisfy Walras' law, and have a substitution matrix $S(p, w)$ that is symmetric and negative semidefinite (n.s.d.) at all (p, w). We now pose the reverse question: *If we observe a demand function $x(p, w)$ that has these properties, can we find preferences that rationalize $x(\cdot)$?* As we show in this section (albeit somewhat unrigorously), the answer is yes; these conditions are sufficient for the existence of rational generating preferences. This problem, known as the *integrability problem*, has a long tradition in economic theory, beginning with Antonelli (1886); we follow the approach of Hurwicz and Uzawa (1971).

There are several theoretical and practical reasons why this question and result are of interest.

On a theoretical level, the result tells us two things. First, it tells us that not only are the properties of homogeneity of degree zero, satisfaction of Walras' law, and a symmetric and negative semidefinite substitution matrix necessary consequences of the preference-based demand theory, but these are also *all* of its consequences. As long as consumer demand satisfies these properties, there is *some* rational preference relation that could have generated this demand.

Second, the result completes our study of the relation between the preference-based theory of demand and the choice-based theory of demand built on the weak axiom. We have already seen, in Section 2.F, that although a rational preference relation always generates demand possessing a symmetric substitution matrix, the weak axiom need not do so. Therefore, we already know that when $S(p, w)$ is not symmetric, demand satisfying the weak axiom cannot be rationalized by preferences. The result studied here tightens this relationship by showing that demand satisfying the weak axiom (plus homogeneity of degree zero and Walras' law) can be rationalized by preferences *if and only if* it has a symmetric substitution matrix $S(p, w)$. Hence, the *only* thing added to the properties of demand by the rational preference hypothesis, beyond what is implied by the weak axiom, homogeneity of degree zero, and Walras' law, is symmetry of the substitution matrix.

On a practical level, the result is of interest for at least two reasons. First, as we shall discuss in Section 3.J, to draw conclusions about welfare effects we need to know the consumer's preferences (or, at the least, her expenditure function). The result tells how and when we can recover this information from observation of the consumer's demand behavior.

Second, when conducting empirical analyses of demand, we often wish to estimate demand functions of a relatively simple form. If we want to allow only functions that can be tied back to an underlying preference relation, there are two ways to do this. One is to specify various utility functions and derive the demand functions that they lead to until we find one that seems statistically tractable. However, the result studied here gives us an easier way; it allows us instead to begin by specifying a tractable demand function and then simply check whether it satisfies the necessary and sufficient conditions that we identify in this section. We do not need to actually derive the utility function; the result allows us to check whether it is, in principle, possible to do so.

The problem of recovering preferences $\succsim$ from $x(p, w)$ can be subdivided into two parts: (i) recovering an expenditure function $e(p, u)$ from $x(p, w)$, and (ii) recovering preferences from the expenditure function $e(p, u)$. Because it is the more straightforward of the two tasks, we discuss (ii) first.

Recovering Preferences from the Expenditure Function

Suppose that $e(p, u)$ is the consumer's expenditure function. By Proposition 3.E.2, it is strictly increasing in u and is continuous, nondecreasing, homogeneous of degree one, and concave in p. In addition, because we are assuming that demand is single-valued, we know that $e(p, u)$ must be differentiable (by Propositions 3.F.1 and 3.G.1).

Given this function $e(p, u)$, how can we recover a preference relation that generates it? Doing so requires finding, for each utility level u, an at-least-as-good-as set $V_u \subset \mathbb{R}^L$ such that $e(p, u)$ is the minimal expenditure required for the consumer to purchase a bundle in V_u at prices $p \gg 0$. That is, we want to identify a set V_u such that, for all

$p \gg 0$, we have

$$e(p, u) = \underset{x \geq 0}{\text{Min}} \quad p \cdot x$$
$$\text{s.t. } x \in V_u.$$

In the framework of Section 3.F, V_u is a set whose support function is precisely $e(p, u)$.

The result in Proposition 3.H.1 shows that the set $V_u = \{x \in \mathbb{R}^L_+ : p \cdot x \geq e(p, u) \text{ for all } p \gg 0\}$ accomplishes this objective.

Proposition 3.H.1: Suppose that $e(p, u)$ is strictly increasing in u and is continuous, increasing, homogeneous of degree one, concave, and differentiable in p. Then, for every utility level u, $e(p, u)$ is the expenditure function associated with the at-least-as-good-as set

$$V_u = \{x \in \mathbb{R}^L_+ : p \cdot x \geq e(p, u) \text{ for all } p \gg 0\}.$$

That is, $e(p, u) = \text{Min}\,\{p \cdot x : x \in V_u\}$ for all $p \gg 0$.

Proof: The properties of $e(p, u)$ and the definition of V_u imply that V_u is nonempty, closed, and bounded below. Given $p \gg 0$, it can be shown that these conditions insure that $\text{Min}\,\{p \cdot x : x \in V_u\}$. exists. It is immediate from the definition of V_u that $e(p, u) \leq \text{Min}\,\{p \cdot x : x \in V_u\}$. What remains in order to establish the result is to show equality. We do this by showing that $e(p, u) \geq \text{Min}\,\{p \cdot x : x \in V_u\}$.

For any p and p', the concavity of $e(p, u)$ in p implies that (see Section M.C of the Mathematical Appendix)

$$e(p', u) \leq e(p, u) + \nabla_p e(p, u) \cdot (p' - p).$$

Because $e(p, u)$ is homogeneous of degree one in p, Euler's formula tells us that $e(p, u) = p \cdot \nabla_p e(p, u)$. Thus, $e(p', u) \leq p' \cdot \nabla_p e(p, u)$ for all p'. But since $\nabla_p e(p, u) \geq 0$, this means that $\nabla_p e(p, u) \in V_u$. It follows that $\text{Min}\,\{p \cdot x : x \in V_u\} \leq p \cdot \nabla_p e(p, u) = e(p, u)$, as we wanted (the last equality uses Euler's formula once more). This establishes the result. ■

Given Proposition 3.H.1, we can construct a set V_u for each level of u. Because $e(p, u)$ is strictly increasing in u, it follows that if $u' > u$, then V_u strictly contains $V_{u'}$. In addition, as noted in the proof of Proposition 3.H.1, each V_u is closed, convex, and bounded below. These various at-least-as-good-as sets then define a preference relation $\succsim$ that has $e(p, u)$ as its expenditure function (see Figure 3.H.1).

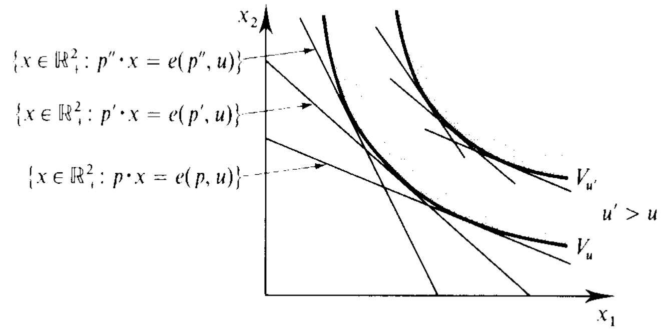


Figure 3.H.1 Recovering preferences from the expenditure function.

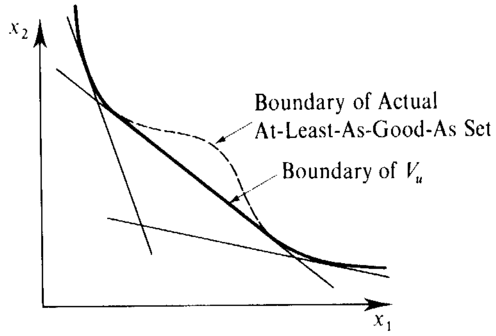


Figure 3.H.2
Recovering preferences from the expenditure function when the consumers' preferences are nonconvex.

Proposition 3.H.1 remains valid, with substantially the same proof, when $e(p, u)$ is not differentiable in p. The preference relation constructed as in the proof of the proposition provides a convex preference relation that generates $e(p, u)$. However, it could happen that there are also nonconvex preferences that generate $e(p, u)$. Figure 3.H.2 illustrates a case where the consumer's actual at-least-as-good-as set is nonconvex. The boundary of this set is depicted with a dashed curve. The solid curve shows the boundary of the set $V_u = \{x \in \mathbb{R}^L_+ : p \cdot x \geq e(p, u)$ for all $p \gg 0\}$. Formally, this set is the convex hull of the consumer's actual at-least-as-good-as set, and it also generates the expenditure function $e(p, u)$.

If $e(p, u)$ is differentiable, then any preference relation that generates $e(p, u)$ must be convex. If it were not, then there would be some utility level u and price vector $p \gg 0$ with several expenditure minimizers (see Figure 3.H.2). At this price-utility pair, the expenditure function would not be differentiable in p.

Recovering the Expenditure Function from Demand

It remains to recover $e(p, u)$ from observable consumer behavior summarized in the Walrasian demand $x(p, w)$. We now discuss how this task (which is, more properly, the actual "integrability problem") can be done. We assume throughout that $x(p, w)$ satisfies Walras' law and homogeneity of degree zero and that it is single-valued.

Let us first consider the case of two commodities ($L = 2$). We normalize $p_2 = 1$. Pick an arbitrary price-wealth point $(p_1^0, 1, w^0)$ and assign a utility value of u^0 to bundle $x(p_1^0, 1, w^0)$. We will now recover the value of the expenditure function $e(p_1, 1, u^0)$ at all prices $p_1 > 0$. Because compensated demand is the derivative of the expenditure function with respect to prices (Proposition 3.G.1), recovering $e(\cdot)$ is equivalent to being able to solve (to "integrate") a differential equation with the independent variable p_1 and the dependent variable e. Writing $e(p_1) = e(p_1, 1, u^0)$ and $x_1(p_1, w) = x_1(p_1, 1, w)$ for simplicity, we need to solve the differential equation,

$$\frac{de(p_1)}{dp_1} = x_1(p_1, e(p_1)), \tag{3.H.1}$$

with the initial condition[18] $e(p_1^0) = w^0$.

If $e(p_1)$ solves (3.H.1) for $e(p_1^0) = w^0$, then $e(p_1)$ is the expenditure function associated with the level of utility u^0. Note, in particular, that if the substitution

18. Technically, (3.H.1) is a nonautonomous system in the (p_1, e) plane. Note that p_1 plays the role of the "t" variable.

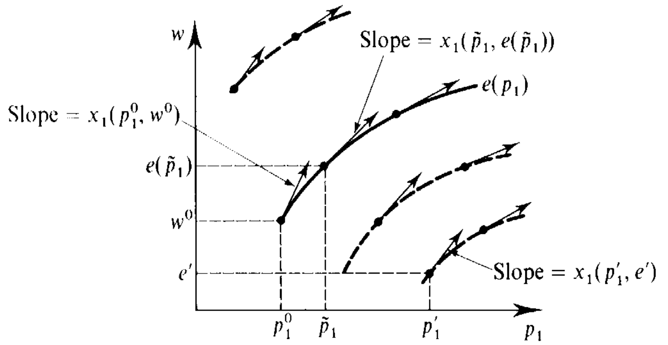


Figure 3.H.3 Recovering the expenditure functions from $x(p, w)$.

matrix is negative semidefinite then $e(p_1)$ will have all the properties of an expenditure function (with the price of good 2 normalized to equal 1). First, because it is the solution to a differential equation, it is by construction continuous in p_1. Second, since $x_1(p, w) \geq 0$, equation (3.H.1) implies that $e(p_1)$ is nondecreasing in p_1. Third, differentiating equation (3.H.1) tells us that

$$\begin{aligned}\frac{d^2e(p_1)}{dp_1^2} &= \frac{\partial x_1(p_1, 1, e(p_1))}{\partial p_1} + \frac{\partial x_1(p_1, 1, e(p_1))}{\partial w} x_1(p_1, 1, e(p_1)) \\ &= s_{11}(p_1, 1, e(p_1)) \leq 0,\end{aligned}$$

so that the solution $e(p_1)$ is concave in p_1.

Solving equation (3.H.1) is a straightforward problem in ordinary differential equations that, nonetheless, we will not go into. A few weak regularity assumptions guarantee that a solution to (3.H.1) exists for any initial condition (p_1^0, w^0). Figure 3.H.3 describes the essence of what is involved: At each price level p_1 and expenditure level e, we are given a direction of movement with slope $x_1(p_1, e)$. For the initial condition (p_1^0, w^0), the graph of $e(p_1)$ is the curve that starts at (p_1^0, w^0) and follows the prescribed directions of movement.

For the general case of L commodities, the situation becomes more complicated. The (ordinary) differential equation (3.H.1) must be replaced by the system of partial differential equations:

$$\begin{aligned}\frac{\partial e(p)}{\partial p_1} &= x_1(p, e(p)) \\ &\vdots \\ \frac{\partial e(p)}{\partial p_L} &= x_L(p, e(p))\end{aligned} \tag{3.H.2}$$

for initial conditions p^0 and $e(p^0) = w^0$. The existence of a solution to (3.H.2) is *not* automatically guaranteed when $L > 2$. Indeed, if there is a solution $e(p)$, then its Hessian matrix $D_p^2 e(p)$ must be symmetric because the Hessian matrix of any twice continuously differentiable function is symmetric. Differentiating equations (3.H.2), which can be written as $\nabla_p e(p) = x(p, e(p))$, tells us that

$$\begin{aligned}D_p^2 e(p) &= D_p x(p, e(p)) + D_w x(p, e(p)) x(p, e(p))^{\mathrm{T}} \\ &= S(p, e(p)).\end{aligned}$$

Therefore, a necessary condition for the existence of a solution is the symmetry of the Slutsky matrix of $x(p, w)$. This is a comforting fact because we know from previous sections that if market demand is generated from preferences, then the Slutsky matrix is indeed symmetric. It turns out that symmetry of $S(p, w)$ is also sufficient for recovery of the consumer's expenditure function. A basic result of the theory of partial differential equations (called *Frobenius' theorem*) tells us that the symmetry of the $L \times L$ derivative matrix of (3.H.2) at all points of its domain is the necessary and sufficient condition for the existence of a solution to (3.H.2). In addition, if a solution $e(p_1, u_0)$ does exist, then, as long as $S(p, w)$ is negative semidefinite, it will possess the properties of an expenditure function.

We therefore conclude that *the necessary and sufficient condition for the recovery of an underlying expenditure function is the symmetry and negative semidefiniteness of the Slutsky matrix.*[19] Recall from Section 2.F that a differentiable demand function satisfying the weak axiom, homogeneity of degree zero, and Walras' law necessarily has a negative semidefinite Slutsky matrix. Moreover, when $L = 2$, the Slutsky matrix is necessarily symmetric (recall Exercise 2.F.12). Thus, for the case where $L = 2$, we can always find preferences that rationalize any differentiable demand function satisfying these three properties. When $L > 2$, however, the Slutsky matrix of a demand function satisfying the weak axiom (along with homogeneity of degree zero and Walras' law) need not be symmetric; preferences that rationalize a demand function satisfying the weak axiom exist only when it is.

Observe that once we know that $S(p, w)$ is symmetric at all (p, w), we can in fact use (3.H.1) to solve (3.H.2). Suppose that with initial conditions p^0 and $e(p^0) = w^0$, we want to recover $e(\bar{p})$. By changing prices one at a time, we can decompose this problem into L subproblems where only one price changes at each step. Say it is price ℓ. Then with p_k fixed for $k \neq \ell$, the ℓth equation of (3.H.2) is an equation of the form (3.H.1), with the subscript 1 replaced by ℓ. It can be solved by the methods appropriate to (3.H.1). Iterating for different goods, we eventually get to $e(\bar{p})$. It is worthwhile to point out that this method makes mechanical sense even if $S(p, w)$ is not symmetric. However, if $S(p, w)$ is not symmetric (and therefore *cannot* be associated with an underlying preference relation and expenditure function), then the value of $e(\bar{p})$ will *depend on the particular path followed from p^0 to $\bar{p}$* (i.e., on which price is raised first). By this absurdity, the mathematics manage to keep us honest!

3.I Welfare Evaluation of Economic Changes

Up to this point, we have studied the preference-based theory of consumer demand from a positive (behavioral) perspective. In this section, we investigate the normative side of consumer theory, called *welfare analysis*. Welfare analysis concerns itself with the evaluation of the effects of changes in the consumer's environment on her well-being.

Although many of the positive results in consumer theory could also be deduced using an approach based on the weak axiom (as we did in Section 2.F), the preference-based approach to consumer demand is of critical importance for welfare

19. This is subject to minor technical requirements.

analysis. Without it, we would have no means of evaluating the consumer's level of well-being.

In this section, we consider a consumer with a rational, continuous, and locally nonsatiated preference relation $\succsim$. We assume, whenever convenient, that the consumer's expenditure and indirect utility functions are differentiable.

We focus here on the welfare effect of a price change. This is only an example, albeit a historically important one, in a broad range of possible welfare questions one might want to address. We assume that the consumer has a fixed wealth level $w > 0$ and that the price vector is initially p^0. We wish to evaluate the impact on the consumer's welfare of a change from p^0 to a new price vector p^1. For example, some government policy that is under consideration, such as a tax, might result in this change in market prices.[20]

Suppose, to start, that we know the consumer's preferences $\succsim$. For example, we may have derived $\succsim$ from knowledge of her (observable) Walrasian demand function $x(p, w)$, as discussed in Section 3.H. If so, it is a simple matter to determine whether the price change makes the consumer better or worse off: if $v(p, w)$ is any indirect utility function derived from $\succsim$, the consumer is worse off if and only if $v(p^1, w) - v(p^0, w) < 0$.

Although any indirect utility function derived from $\succsim$ suffices for making this comparison, one class of indirect utility functions deserves special mention because it leads to measurement of the welfare change expressed in dollar units. These are called *money metric* indirect utility functions and are constructed by means of the expenditure function. In particular, starting from any indirect utility function $v(\cdot, \cdot)$, choose an arbitrary price vector $\bar{p} \gg 0$, and consider the function $e(\bar{p}, v(p, w))$. This function gives the wealth required to reach the utility level $v(p, w)$ when prices are $\bar{p}$. Note that this expenditure is strictly increasing as a function of the level $v(p, w)$, as shown in Figure 3.I.1. Thus, viewed as a function of (p, w), $e(\bar{p}, v(p, w))$ is itself an indirect utility function for $\succsim$, and

$$e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w))$$

provides a measure of the welfare change expressed in dollars.[21]

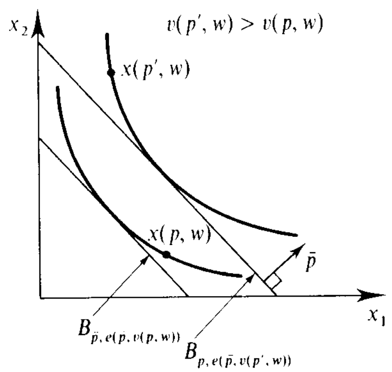


Figure 3.I.1
A money metric indirect utility function.

20. For the sake of expositional simplicity, we do not consider changes that affect wealth here. However, the analysis readily extends to that case (see Exercise 3.I.12).

21. Note that this measure is unaffected by the choice of the initial indirect utility function $v(p, w)$; it depends only on the consumer's preferences $\succsim$ (see Figure 3.I.1).

A money metric indirect utility function can be constructed in this manner for any price vector $\bar{p} \gg 0$. Two particularly natural choices for the price vector $\bar{p}$ are the initial price vector p^0 and the new price vector p^1. These choices lead to two well-known measures of welfare change originating in Hicks (1939), the *equivalent variation* (EV) and the *compensating variation* (CV). Formally, letting $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$, and noting that $e(p^0, u^0) = e(p^1, u^1) = w$, we define

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w \tag{3.I.1}$$

and

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0). \tag{3.I.2}$$

The equivalent variation can be thought of as the dollar amount that the consumer would be indifferent about accepting in lieu of the price change; that is, it is the change in her wealth that would be *equivalent* to the price change in terms of its welfare impact (so it is negative if the price change would make the consumer worse off). In particular, note that $e(p^0, u^1)$ is the wealth level at which the consumer achieves exactly utility level u^1, the level generated by the price change, at prices p^0. Hence, $e(p^0, u^1) - w$ is the net change in wealth that causes the consumer to get utility level u^1 at prices p^0. We can also express the equivalent variation using the indirect utility function $v(\cdot, \cdot)$ in the following way: $v(p^0, w + EV) = u^1$.[22]

The compensating variation, on the other hand, measures the net revenue of a planner who must *compensate* the consumer for the price change after it occurs, bringing her back to her original utility level u^0. (Hence, the compensating variation is negative if the planner would have to pay the consumer a positive level of compensation because the price change makes her worse off.) It can be thought of as the negative of the amount that the consumer would be just willing to accept from the planner to allow the price change to happen. The compensating variation can also be expressed in the following way: $v(p^1, w - CV) = u^0$.

Figure 3.I.2 depicts the equivalent and compensating variation measures of welfare change. Because both the EV and the CV correspond to measurements of the changes in a money metric indirect utility function, both provide a correct welfare ranking of the alternatives p^0 and p^1; that is, the consumer is better off under p^1 if and only if these measures are positive. In general, however, the specific dollar

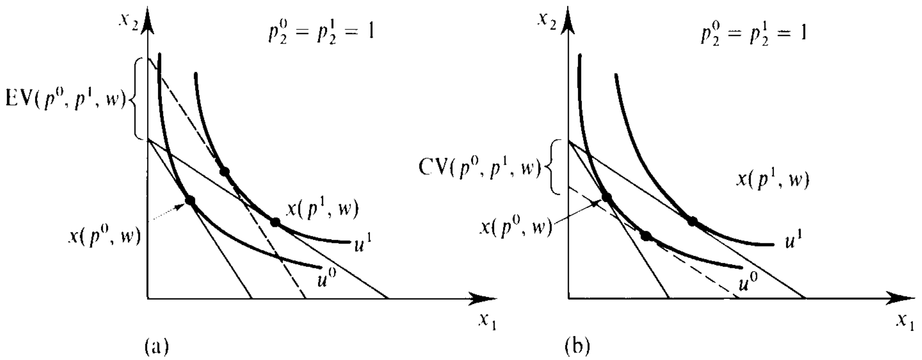


Figure 3.I.2
The equivalent (a) and compensating (b) variation measures of welfare change.

22. Note that if $u^1 = v(p^0, w + EV)$, then $e(p^0, u^1) = e(p^0, v(p^0, w + EV)) = w + EV$. This leads to (3.I.1).

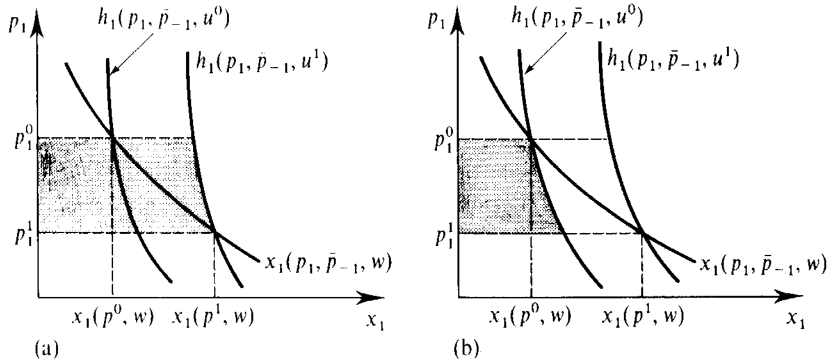


Figure 3.I.3
(a) The equivalent variation.
(b) The compensating variation.

amounts calculated using the EV and CV measures will differ because of the differing price vectors at which compensation is assumed to occur in these two measures of welfare change.

The equivalent and compensating variations have interesting representations in terms of the Hicksian demand curve. Suppose, for simplicity, that only the price of good 1 changes, so that $p_1^0 \neq p_1^1$ and $p_\ell^0 = p_\ell^1 = \bar{p}_\ell$ for all $\ell \neq 1$. Because $w = e(p^0, u^0) = e(p^1, u^1)$ and $h_1(p, u) = \partial e(p, u)/\partial p_1$, we can write

$$\begin{aligned} EV(p^0, p^1, w) &= e(p^0, u^1) - w \\ &= e(p^0, u^1) - e(p^1, u^1) \\ &= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1, \end{aligned} \tag{3.I.3}$$

where $\bar{p}_{-1} = (p_2, \ldots, p_L)$. Thus, the change in consumer welfare as measured by the equivalent variation can be represented by the area lying between p_1^0 and p_1^1 and to the left of the Hicksian demand curve for good 1 associated with utility level u^1 (it is equal to this area if $p_1^1 < p_1^0$ and is equal to its negative if $p_1^1 > p_1^0$). The area is depicted as the shaded region in Figure 3.I.3(a).

Similarly, the compensating variation can be written as

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1. \tag{3.I.4}$$

Note that we now use the initial utility level u^0. See Figures 3.I.3(b) for its graphic representation.

Figure 3.I.3 depicts a case where good 1 is a normal good. As can be seen in the figure, when this is so, we have $EV(p^0, p^1, w) > CV(p^0, p^1, w)$ (you should check that the same is true when $p_1^1 > p_1^0$). This relation between the EV and the CV reverses when good 1 is inferior (see Exercise 3.I.3). However, if there is no wealth effect for good 1 (e.g., if the underlying preferences are quasilinear with respect to some good $\ell \neq 1$), the CV and EV measures are the *same* because we then have

$$h_1(p_1, \bar{p}_{-1}, u^0) = x_1(p_1, \bar{p}_{-1}, w) = h_1(p_1, \bar{p}_{-1}, u^1).$$

In this case of no wealth effects, we call the common value of CV and EV, which is also the value of the area lying between p_1^0 and p_1^1 and to the left of the market (i.e., Walrasian) demand curve for good 1, the change in *Marshallian consumer surplus.*[23]

23. The term originates from Marshall (1920), who used the area to the left of the market demand curve as a welfare measure in the special case where wealth effects are absent.

Exercise 3.I.1: Suppose that the change from price vector p^0 to price vector p^1 involves a change in the prices of both good 1 (from p_1^0 to p_1^1) and good 2 (from p_2^0 to p_2^1). Express the equivalent variation in terms of the sum of integrals under appropriate Hicksian demand curves for goods 1 and 2. Do the same for the compensating variation measure. Show also that if there are no wealth effects for either good, the compensating and equivalent variations are equal.

Example 3.I.1: *The Deadweight Loss from Commodity Taxation.* Consider a situation where the new price vector p^1 arises because the government puts a tax on some commodity. To be specific, suppose that the government taxes commodity 1, setting a tax on the consumer's purchases of good 1 of t per unit. This tax changes the effective price of good 1 to $p_1^1 = p_1^0 + t$ while prices for all other commodities $\ell \neq 1$ remain fixed at p_ℓ^0 (so we have $p_\ell^1 = p_\ell^0$ for all $\ell \neq 1$). The total revenue raised by the tax is therefore $T = tx_1(p^1, w)$.

An alternative to this commodity tax that raises the same amount of revenue for the government without changing prices is imposition of a "lump-sum" tax of T directly on the consumer's wealth. Is the consumer better or worse off facing this lump-sum wealth tax rather than the commodity tax? She is worse off under the commodity tax if the equivalent variation of the commodity tax $EV(p^0, p^1, w)$, which is negative, is less than $-T$, the amount of wealth she will lose under the lump-sum tax. Put in terms of the expenditure function, this says that she is worse off under commodity taxation if $w - T > e(p^0, u^1)$, so that her wealth after the lump-sum tax is greater than the wealth level that is required at prices p^0 to generate the utility level that she gets under the commodity tax, u^1. The difference $(-T) - EV(p^0, p^1, w) = w - T - e(p^0, u^1)$ is known as the *deadweight loss of commodity taxation.* It measures the extra amount by which the consumer is made worse off by commodity taxation above what is necessary to raise the same revenue through a lump-sum tax.

The deadweight loss measure can be represented in terms of the Hicksian demand curve at utility level u^1. Since $T = tx_1(p^1, w) = th_1(p^1, u^1)$, we can write the deadweight loss as follows [we again let $\bar{p}_{-1} = (\bar{p}_2, \ldots, \bar{p}_L)$, where $p_\ell^0 = p_\ell^1 = \bar{p}_\ell$ for all $\ell \neq 1$]:

$$\begin{aligned}(-T) - EV(p^0, p^1, w) &= e(p^1, u^1) - e(p^0, u^1) - T \\ &= \int_{p_1^0}^{p_1^0 + t} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1 - th_1(p_1^0 + t, \bar{p}_{-1}, u^1) \\ &= \int_{p_1^0}^{p_1^0 + t} [h_1(p_1, \bar{p}_{-1}, u^1) - h_1(p_1^0 + t, \bar{p}_{-1}, u^1)]\, dp_1. \qquad (3.I.5)\end{aligned}$$

Because $h_1(p, u)$ is nonincreasing in p_1, this expression (and therefore the deadweight loss of taxation) is nonnegative, and it is strictly positive if $h_1(p, u)$ is strictly decreasing in p_1. In Figure 3.I.4(a), the deadweight loss is depicted as the area of the crosshatched triangular region. This region is sometimes called the *deadweight loss triangle.*

This deadweight loss measure can also be represented in the commodity space. For example, suppose that $L = 2$, and normalize $p_2^0 = 1$. Consider Figure 3.I.5. Since $(p_1^0 + t)x_1(p^1, w) + p_2^0 x_2(p^1, w) = w$, the bundle $x(p^1, w)$ lies not only on the budget line associated with budget set $B_{p^1, w}$ but also on the budget line associated with budget set $B_{p^0, w-T}$. In contrast, the budget set that generates a utility of u^1 for the consumer at prices p^0 is $B_{p^0, e(p^0, u^1)}$ (or, equivalently,

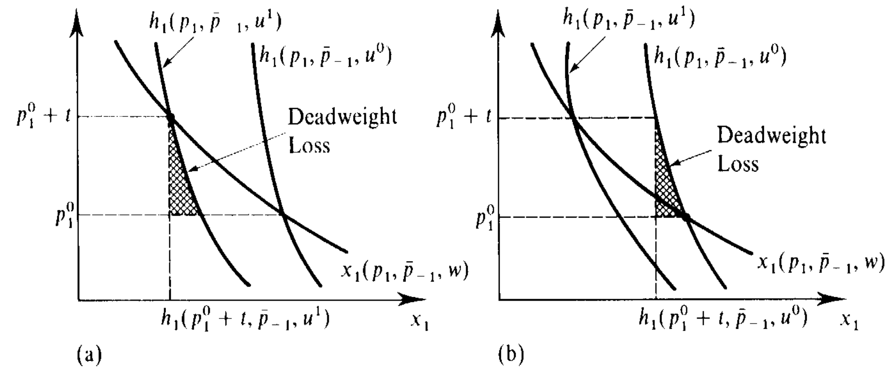


Figure 3.I.4
The deadweight loss from commodity taxation.
(a) Measure based at u^1.
(b) Measure based at u^0.

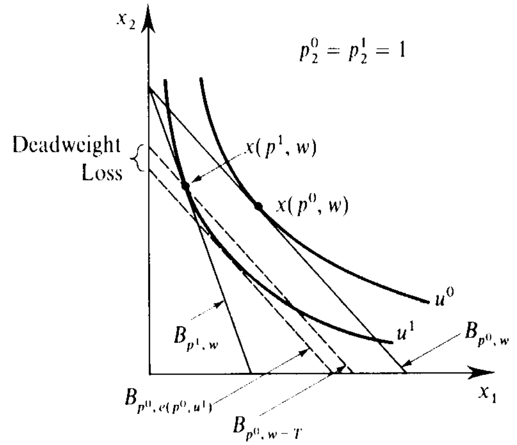


Figure 3.I.5
An alternative depiction of the deadweight loss from commodity taxation.

$B_{p^0,w+EV}$). The deadweight loss is the vertical distance between the budget lines associated with budget sets $B_{p^0,w-T}$ and $B_{p^0,e(p^0,u^1)}$ (recall that $p_2^0 = 1$).

A similar deadweight loss triangle can be calculated using the Hicksian demand curve $h_1(p, u^0)$. It also measures the loss from commodity taxation, but in a different way. In particular, suppose that we examine the surplus or deficit that would arise if the government were to compensate the consumer to keep her welfare under the tax equal to her pretax welfare u^0. The government would run a deficit if the tax collected $th_1(p^1, u^0)$ is less than $-CV(p^0, p^1, w)$ or, equivalently, if $th_1(p^1, u^0) < e(p^1, u^0) - w$. Thus, the deficit can be written as

$$\begin{aligned} -CV(p^0, p^1, w) - th_1(p^1, u) &= e(p^1, u^0) - e(p^0, u^0) - th_1(p^1, u^0) \\ &= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1 - th_1(p_1^0 + t, \bar{p}_{-1}, u^0) \\ &= \int_{p_1^0}^{p_1^0+t} [h_1(p_1, \bar{p}_{-1}, u^0) - h_1(p_1^0 + t, \bar{p}_{-1}, u^0)]\, dp_1 . \end{aligned} \tag{3.I.6}$$

which is again strictly positive as long as $h_1(p, u)$ is strictly decreasing in p_1. This deadweight loss measure is equal to the area of the crosshatched triangular region in Figure 3.I.4(b). ■

Exercise 3.I.2: Calculate the derivative of the deadweight loss measures (3.I.5) and (3.I.6) with respect to t. Show that, evaluated at $t = 0$, these derivatives are equal to zero but that if $h_1(p, u^0)$ is strictly decreasing in p_1, they are strictly positive at all $t > 0$. Interpret.

Up to now, we have considered only the question of whether the consumer was better off at p^1 than at the initial price vector p^0. We saw that both EV and CV provide a correct welfare ranking of p^0 and p^1. Suppose, however, that p^0 is being compared with two possible price vectors p^1 and p^2. In this case, p^1 is better than p^2 if and only if $EV(p^0, p^1, w) > EV(p^0, p^2, w)$, since

$$EV(p^0, p^1, w) - EV(p^0, p^2, w) = e(p^0, u^1) - e(p^0, u^2).$$

Thus, the EV measures $EV(p^0, p^1, w)$ and $EV(p^0, p^2, w)$ can be used not only to compare these two price vectors with p^0 but also to determine which of them is better for the consumer. A comparison of the compensating variations $CV(p^0, p^1, w)$ and $CV(p^0, p^2, w)$, however, will not necessarily rank p^1 and p^2 correctly. The problem is that the CV measure uses the new prices as the base prices in the money metric indirect utility function, using p^1 to calculate $CV(p^0, p^1, w)$ and p^2 to calculate $CV(p^0, p^2, w)$. So

$$CV(p^0, p^1, w) - CV(p^0, p^2, w) = e(p^2, u^0) - e(p^1, u^0),$$

which need not correctly rank p^1 and p^2 [see Exercise 3.I.4 and Chipman and Moore (1980)]. In other words, fixing p^0, $EV(p^0, \cdot, w)$ is a valid indirect utility function (in fact, a money metric one), but $CV(p^0, \cdot, w)$ is not.[24]

An interesting example of the comparison of several possible new price vectors arises when a government is considering which goods to tax. Suppose, for example, that two different taxes are being considered that could raise tax revenue of T: a tax on good 1 of t_1 (creating new price vector p^1) and a tax on good 2 of t_2 (creating new price vector p^2). Note that since they raise the same tax revenue, we have $t_1 x_1(p^1, w) = t_2 x_2(p^2, w) = T$ (see Figure 3.I.6). Because tax t_1

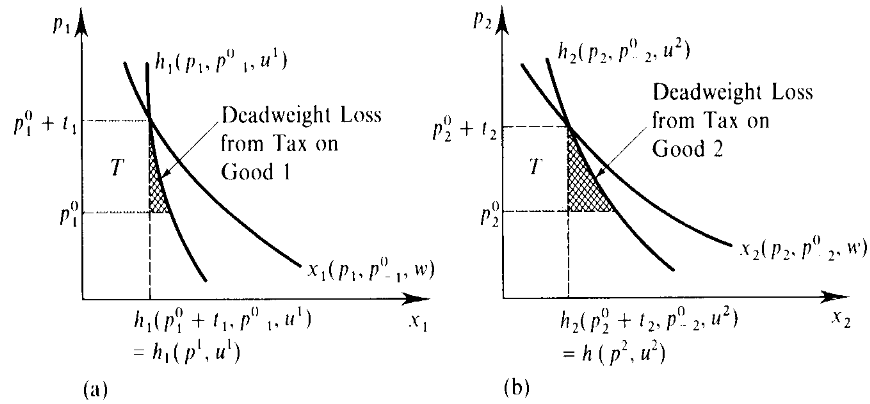


Figure 3.I.6
Comparing two taxes that raise revenue T.
(a) Tax on good 1.
(b) Tax on good 2.

is better than tax t_2 if and only if $EV(p^0, p^1, w) > EV(p^0, p^2, w)$, t_1 is better than t_2 if and only if $[(-T) - EV(p^0, p^1, w)] < [(-T) - EV(p^0, p^2, w)]$, that is, if and only if the deadweight loss arising under tax t_1 is less than that arising under tax t_2.

---

24. Of course, we can rank p^1 and p^2 correctly by seeing whether $CV(p^1, p^2, w)$ is positive or negative.

In summary, if we know the consumer's expenditure function, we can precisely measure the welfare impact of a price change; moreover, we can do it in a convenient way (in dollars). In principle, this might well be the end of the story because, as we saw in Section 3.H, we can recover the consumer's preferences and expenditure function from the observable Walrasian demand function $x(p, w)$.[25] Before concluding, however, we consider two further issues. We first ask whether we may be able to say anything about the welfare effect of a price change when we *do not* have enough information to recover the consumer's expenditure function. We describe a test that provides a sufficient condition for the consumer's welfare to increase from the price change and that uses information only about the two price vectors p^0, p^1 and the initial consumption bundle $x(p^0, w)$. We then conclude by discussing in detail the extent to which the welfare change can be approximated by means of the area to the left of the market (Walrasian) demand curve, a topic of significant historical importance.

Welfare Analysis with Partial Information

In some circumstances, we may not be able to derive the consumer's expenditure function because we may have only limited information about her Walrasian demand function. Here we consider what can be said when the *only* information we possess is knowledge of the two price vectors p^0, p^1 and the consumer's initial consumption bundle $x^0 = x(p^0, w)$. We begin, in Proposition 3.I.1, by developing a simple sufficiency test for whether the consumer's welfare improves as a result of the price change.

Proposition 3.I.1: Suppose that the consumer has a locally nonsatiated rational preference relation $\succsim$. If $(p^1 - p^0)\cdot x^0 < 0$, then the consumer is strictly better off under price wealth situation (p^1, w) than under (p^0, w).

Proof: The result follows simply from revealed preference. Since $p^0\cdot x^0 = w$ by Walras' law, if $(p^1 - p^0)\cdot x^0 < 0$, then $p^1\cdot x^0 < w$. But if so, x^0 is still affordable under prices p^1 and is, moreover, in the interior of budget set $B_{p^1,w}$. By local nonsatiation, there must therefore be a consumption bundle in $B_{p^1,w}$ that the consumer strictly prefers to x^0. ■

The test in Proposition 3.I.1 can be viewed as a first-order approximation to the true welfare change. To see this, take a first-order Taylor expansion of $e(p, u)$ around the initial prices p^0:

$$e(p^1, u^0) = e(p^0, u^0) + (p^1 - p^0)\cdot\nabla_p e(p^0, u^0) + o(\|p^1 - p^0\|). \qquad (3.I.7)$$

If $(p^1 - p^0)\cdot\nabla_p e(p^0, u^0) < 0$ and the second-order remainder term could be ignored, we would have $e(p^1, u^0) < e(p^0, u^0) = w$, and so we could conclude that the consumer's welfare is greater after the price change. But the concavity of $e(\cdot, u^0)$ in p implies that the remainder term is nonpositive. Therefore, ignoring the remainder term leads to no error here; we do have $e(p^1, u^0) < w$ if $(p^1 - p^0)\cdot\nabla_p e(p^0, u^0) < 0$. Using Proposition 3.G.1 then tells us that $(p^1 - p^0)\cdot\nabla_p e(p^0, u^0) = (p^1 - p^0)\cdot h(p^0, u^0) = (p^1 - p^0)\cdot x^0$, and so we get exactly the test in Proposition 3.I.1.

25. As a practical matter, in applications you should use whatever are the state-of-the-art techniques for performing this recovery.

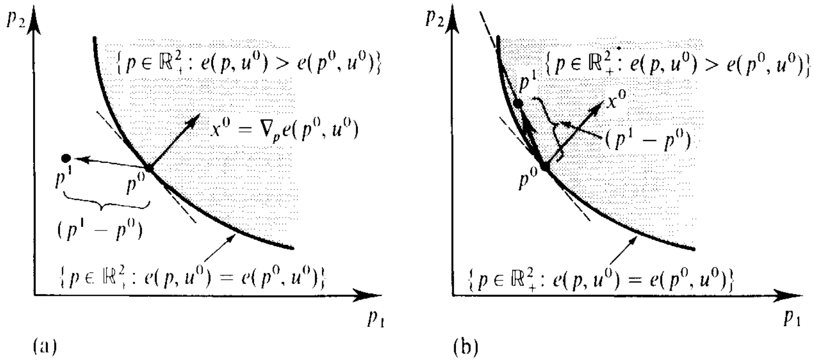


Figure 3.I.7
The welfare test of Propositions 3.I.1 and 3.I.2.
(a) $(p^1 - p^0)\cdot x^0 < 0$.
(b) $(p^1 - p^0)\cdot x^0 > 0$.

What if $(p^1 - p^0)\cdot x^0 > 0$? Can we then say anything about the direction of change in welfare? As a general matter, no. However, examination of the first-order Taylor expansion (3.I.7) tells us that we get a definite conclusion if the price change is, in an appropriate sense, small enough because the remainder term then becomes insignificant relative to the first-order term and can be neglected. This gives the result shown in Proposition 3.I.2.

Proposition 3.I.2: Suppose that the consumer has a differentiable expenditure function. Then if $(p^1 - p^0)\cdot x^0 > 0$, there is a sufficiently small $\bar{\alpha} \in (0, 1)$ such that for all $\alpha < \bar{\alpha}$, we have $e((1-\alpha)p^0 + \alpha p^1, u^0) > w$, and so the consumer is strictly better off under price wealth situation (p^0, w) than under $((1-\alpha)p^0 + \alpha p^1, w)$.

Figure 3.I.7 illustrates these results for the cases where p^1 is such that $(p^1 - p^0)\cdot x^0 < 0$ [panel (a)] and $(p^1 - p^0)\cdot x^0 > 0$ [panel (b)]. In the figure the set of prices $\{p \in \mathbb{R}^2_+ : e(p, u^0) \geq e(p^0, u^0)\}$ is drawn in price space. The concavity of $e(\cdot, u)$ gives it the shape depicted. The initial price vector p^0 lies in this set. By Proposition 3.G.1, the gradient of the expenditure function at this point, $\nabla_p e(p^0, u^0)$, is equal to x^0, the initial consumption bundle. The vector $(p^1 - p^0)$ is the vector connecting point p^0 to the new price point p^1. Figure 3.I.7(a) shows a case where $(p^1 - p^0)\cdot x^0 < 0$. As can be seen there, p^1 lies outside of the set $\{p \in \mathbb{R}^2_+ : e(p, u^0) \geq e(p^0, u^0)\}$, and so we must have $e(p^0, u^0) > e(p^1, u^0)$. In Figure 3.I.7(b), on the other hand, we show a case where $(p^1 - p^0)\cdot x^0 > 0$. Proposition 3.I.2 can be interpreted as asserting that in this case if $(p^1 - p^0)$ is small enough, then $e(p^0, u^0) < e(p^1, u^0)$. This can be seen in Figure 3.I.7(b), because if $(p^1 - p^0)\cdot x^0 > 0$ and p^1 is close enough to p^0 [in the ray with direction $p^1 - p^0$], then price vector p^1 lies in the set $\{p \in \mathbb{R}^2_+ : e(p, u^0) > e(p^0, u^0)\}$.

Using the Area to the Left of the Walrasian (Market) Demand Curve as an Approximate Welfare Measure

Improvements in computational abilities have made the recovery of the consumer's preferences/expenditure function from observed demand behavior, along the lines discussed in Section 3.I, far easier than was previously the case.[26] Traditionally,

26. They have also made it much easier to estimate complicated demand systems that are explicitly derived from utility maximization and from which the parameters of the expenditure function can be derived directly.

however, it has been common practice in applied analyses to rely on approximations of the true welfare change.

We have already seen in (3.I.3) and (3.I.4) that the welfare change induced by a change in the price of good 1 can be exactly computed by using the area to the left of an appropriate Hicksian demand curve. However, these measures present the problem of not being directly observable. A simpler procedure that has seen extensive use appeals to the Walrasian (market) demand curve instead. We call this estimate of welfare change the *area variation* measure (or AV):

$$AV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x_1(p_1, \bar{p}_{-1}, w)\, dp_1. \tag{3.I.8}$$

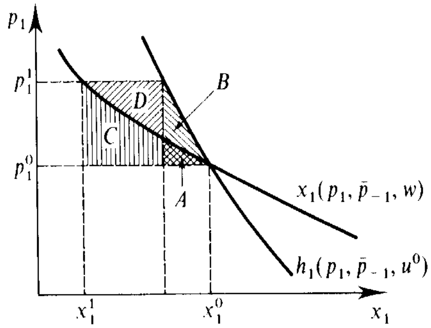
If there are no wealth effects for good 1, then, as we have discussed, $x_1(p, w) = h_1(p, u^0) = h_1(p, u^1)$ for all p and the area variation measure is exactly equal to the equivalent and compensating variation measures. This corresponds to the case studied by Marshall (1920) in which the marginal utility of numeraire is constant. In this circumstance, where the AV measure gives an exact measure of welfare change, the measure is known as the change in *Marshallian consumer surplus.*

More generally, as Figures 3.I.3(a) and 3.I.3(b) make clear, when good 1 is a normal good, the area variation measure overstates the compensating variation and understates the equivalent variation (convince yourself that this is true both when p_1 falls and when p_1 rises). When good 1 is inferior, the reverse relations hold. Thus, when evaluating the welfare change from a change in prices of several goods, or when comparing two different possible price changes, the area variation measure need not give a correct evaluation of welfare change (e.g., see Exercise 3.I.10).

Naturally enough, however, if the wealth effects for the goods under consideration are small, the approximation errors are also small and the area variation measure is almost correct. Marshall argued that if a good is just one commodity among many, then because one extra unit of wealth will spread itself around, the wealth effect for the commodity is bound to be small; therefore, no significant errors will be made by evaluating the welfare effects of price changes for that good using the area measure. This idea can be made precise; for an advanced treatment, see Vives (1987). It is important, however, not to fall into the fallacy of composition; if we deal with a large number of commodities, then while the approximating error may be small for each individually, it may nevertheless not be small in the aggregate.

If $(p_1^1 - p_1^0)$ is small, then the error involved using the area variation measure becomes small as a fraction of the true welfare change. Consider, for example, the compensating variation.[27] In Figure 3.I.8, we see that the area $B + D$, which measures the difference between the area variation and the true compensating variation, becomes small as a fraction of the true compensating variation when $(p_1^1 - p_1^0)$ is small. This might seem to suggest that the area variation measure is a good approximation of the compensating variation measure for small price changes. Note, however, that the same property would hold if instead of the Walrasian demand

27. All the points that follow apply to the equivalent variation as well.



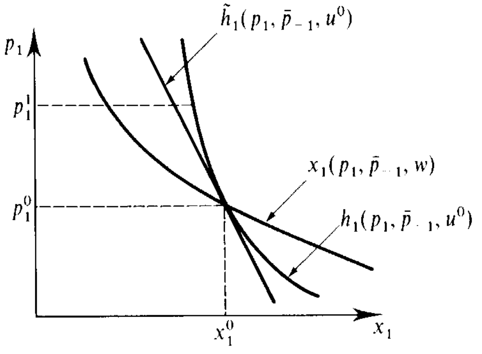


Figure 3.I.8 (left)
The error in using the area variation measure of welfare change.

Figure 3.I.9 (right)
A first-order approximation of $h(p, u^0)$ at p^0.

function we were to use *any* function that takes the value $x_1(p_1^0, p_{-1}^0, w)$ at p_1^0.[28] In fact, the approximation error may be quite large *as a fraction of the deadweight loss* [this point is emphasized by Hausman (1981)]. In Figure 3.I.8, for example, the deadweight loss calculated using the Walrasian demand curve is the area $A + C$, whereas the real one is the area $A + B$. The percentage difference between these two areas need not grow small as the price change grows small.[29]

When $(p_1^1 - p_1^0)$ is small, there is a superior approximation procedure available In particular, suppose we take a first-order Taylor approximation of $h(p, u^0)$ at p^0

$$\tilde{h}(p, u^0) = h(p^0, u^0) + D_p h(p^0, u^0)(p - p^0)$$

and we calculate

$$\int_{p_1^1}^{p_1^0} \tilde{h}_1(p_1, \bar{p}_{-1}, u^0)\, dp_1 \tag{3.I.9}$$

as our approximation of the welfare change. The function $\tilde{h}_1(p_1, \bar{p}_{-1}, u^0)$ is depicted in Figure 3.I.9. As can be seen in the figure, because $\tilde{h}_1(p_1, \bar{p}_{-1}, u^0)$ has the same slope as the true Hicksian demand function $h_1(p, u^0)$ at p^0, for small price changes this approximation comes closer than expression (3.I.8) to the true welfare change (and in contrast with the area variation measure, it provides an adequate approximation to the deadweight loss). Because the Hicksian demand curve is the first derivative of the expenditure function, this first-order expansion of the Hicksian demand function at p^0 is, in essence, a second-order expansion of the expenditure function around p^0. Thus, this approximation can be viewed as the natural extension of the first-order test discussed above; see expression (3.I.7).

The approximation in (3.I.9) is directly computable from knowledge of the observable Walrasian demand function $x_1(p, w)$. To see this, note that because $h(p^0, u^0) = x(p^0, w)$ and $D_p h(p^0, u^0) = S(p^0, w)$, $\tilde{h}(p, u^0)$ can be expressed solely in terms that involve the Walrasian demand function and its derivatives at the point

28. In effect, the property identified here amounts to saying that the Walrasian demand function provides a first-order approximation to the compensating variation. Indeed, note that the derivatives of $CV(p^1, p^0, w)$, $EV(p^1, p^0, w)$, and $AV(p^1, p^0, w)$ with respect to p_1^1 evaluated at p_1^0 are all precisely $x_1(p_1^0, p_{-1}^0, w)$.

29. Thus, for example, in the problem discussed above where we compare the deadweight losses induced by taxes on two different commodities that both raise revenue T, the area variation measure need not give the correct ranking even for small taxes.

(p^0, w):

$$\tilde{h}(p, u^0) = x(p^0, w) + S(p^0, w)(p - p^0).$$

In particular, since only the price of good 1 is changing, we have

$$\tilde{h}_1(p_1, \bar{p}_{-1}, u^0) = x_1(p_1^0, \bar{p}_{-1}, w) + s_{11}(p_1^0, \bar{p}_{-1}, w)(p_1 - p_1^0),$$

where

$$s_{11}(p_1^0, \bar{p}_{-1}, w) = \frac{\partial x_1(p^0, w)}{\partial p_1} + \frac{\partial x_1(p^0, w)}{\partial w} x_1(p^0, w).$$

When $(p^1 - p^0)$ is small, this procedure provides a better approximation to the true compensating variation than does the area variation measure. However, if $(p^1 - p^0)$ is large, we cannot tell which is the better approximation. It is entirely possible for the area variation measure to be superior. After all, its use guarantees some sensitivity of the approximation to demand behavior away from p^0, whereas the use of $\tilde{h}(p, u^0)$ does not.

3.J The Strong Axiom of Revealed Preference

We have seen that in the context of consumer demand theory, consumer choice may satisfy the weak axiom but not be capable of being generated by a rational preference relation (see Sections 2.F and 3.G). We could therefore ask: Can we find a necessary and sufficient consistency condition on consumer demand behavior that is in the same style as the WA but that does imply that demand behavior can be rationalized by preferences? The answer is "yes", and it was provided by Houthakker (1950) in the form of the *strong axiom of revealed preference* (*SA*), a kind of recursive closure of the weak axiom.[30]

Definition 3.J.1: The market demand function $x(p, w)$ satisfies the *strong axiom of revealed preference* (the SA) if for any list

$$(p^1, w^1), \ldots, (p^N, w^N)$$

with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n < N - 1$, we have $p^N \cdot x(p^1, w^1) > w^N$ whenever $p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \leq N - 1$.

In words, if $x(p^1, w^1)$ is *directly or indirectly revealed preferred* to $x(p^N, w^N)$, then $x(p^N, w^N)$ cannot be (directly) revealed preferred to $x(p^1, w^1)$ [so $x(p^1, w^1)$ cannot be affordable at (p^N, w^N)]. For example, the SA was violated in Example 2.F.1. It is clear that the SA is satisfied if demand originates in rational preferences. The converse is a deeper result. It is stated in Proposition 3.J.1; the proof, which is advanced, is presented in small type.

Proposition 3.J.1: If the Walrasian demand function $x(p, w)$ satisfies the strong axiom of revealed preference then there is a rational preference relation $\succsim$ that rationalizes $x(p, w)$, that is, such that for all (p, w), $x(p, w) \succ y$ for every $y \neq x(p, w)$ with $y \in B_{p,w}$.

30. For an informal account of revealed preference theory after Samuelson, see Mas-Colell (1982).

Proof: We follow Richter (1966). His proof is based on set theory and differs markedly from the differential equations techniques used originally by Houthakker.[31]

Define a relation $\succ^1$ on commodity vectors by letting $x \succ^1 y$ whenever $x \neq y$ and we have $x = x(p, w)$ and $p \cdot y \leq w$ for some (p, w). The relation $\succ^1$ can be read as "directly revealed preferred to." From $\succ^1$ define a new relation $\succ^2$, to be read as "directly or indirectly revealed preferred to," by letting $x \succ^2 y$ whenever there is a chain $x^1 \succ^1 x^2 \succ^1, \ldots, \succ^1 x^N$ with $x^1 = x$ and $x^N = y$. Observe that, by construction, $\succ^2$ is transitive. According to the SA, $\succ^2$ is also irreflexive (i.e., $x \succ^2 x$ is impossible). A certain axiom of set theory (known as Zorn's lemma) tells us the following: *Every relation $\succ^2$ that is transitive and irreflexive* (called a *partial order*) *has a total extension* $\succ^3$, an irreflexive and transitive relation such that, first, $x \succ^2 y$ implies $x \succ^3 y$ and, second, whenever $x \neq y$, we have either $x \succ^3 y$ or $y \succ^3 x$. Finally, we can define $\succsim$ by letting $x \succsim y$ whenever $x = y$ or $x \succ^3 y$. It is not difficult now to verify that $\succsim$ is complete and transitive and that $x(p, w) \succ y$ whenever $p \cdot y \leq w$ and $y \neq x(p, w)$. ■

The proof of Proposition 3.J.1 uses only the single-valuedness of $x(p, w)$. Provided choice is single-valued, the same result applies to the abstract theory of choice of Chapter 1. The fact that the budgets are competitive is immaterial.

In Exercise 3.J.1, you are asked to show that the WA is equivalent to the SA when $L = 2$. Hence, by Proposition 3.J.1, when $L = 2$ and demand satisfies the WA, we can always find a rationalizing preference relation, a result that we have already seen in Section 3.H. When $L > 2$, however, the SA is stronger than the WA. In fact, Proposition 3.J.1 tells us that a choice-based theory of demand founded on the strong axiom is essentially equivalent to the preference-based theory of demand presented in this chapter.

The strong axiom is therefore essentially equivalent both to the rational preference hypothesis and to the symmetry and negative semidefiniteness of the Slutsky matrix. We have seen that the weak axiom is essentially equivalent to the negative semidefiniteness of the Slutsky matrix. It is therefore natural to ask whether there is an assumption on preferences that is weaker than rationality and that leads to a theory of consumer demand equivalent to that based on the WA. Violations of the SA mean cycling choice, and violations of the symmetry of the Slutsky matrix generate path dependence in attempts to "integrate back" to preferences. This suggests preferences that may violate the transitivity axiom. See the appendix with W. Shafer in Kihlstrom, Mas-Colell, and Sonnenschein (1976) for further discussion of this point.

APPENDIX A: CONTINUITY AND DIFFERENTIABILITY PROPERTIES OF WALRASIAN DEMAND

In this appendix, we investigate the continuity and differentiability properties of the Walrasian demand correspondence $x(p, w)$. We assume that $x \gg 0$ for all $(p, w) \gg 0$ and $x \in x(p, w)$.

31. Yet a third approach, based on linear programming techniques, was provided by Afriat (1967).

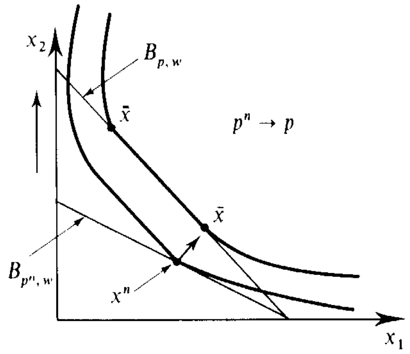


Figure 3.AA.1 An upper hemicontinuous Walrasian demand correspondence.

Continuity

Because $x(p, w)$ is, in general, a correspondence, we begin by introducing a generalization of the more familiar continuity property for functions, called *upper hemicontinuity*.

Definition 3.AA.1: The Walrasian demand correspondence $x(p, w)$ is *upper hemicontinuous* at $(\bar{p}, \bar{w})$ if whenever $(p^n, w^n) \to (\bar{p}, \bar{w})$, $x^n \in x(p^n, w^n)$ for all n, and $x = \lim_{n \to \infty} x^n$, we have $x \in x(p, w)$.[32]

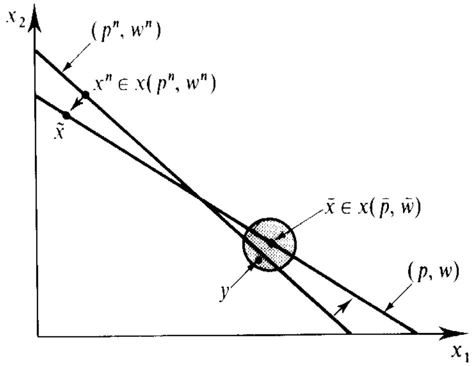
In words, a demand correspondence is upper hemicontinuous at $(\bar{p}, \bar{w})$ if for any sequence of price–wealth pairs the limit of any sequence of optimal demand bundles is optimal (although not necessarily uniquely so) at the limiting price–wealth pair. If $x(p, w)$ is single-valued at all $(p, w) \gg 0$, this notion is equivalent to the usual continuity property for functions.

Figure 3.AA.1 depicts an upper hemicontinuous demand correspondence: When $p^n \to p$, $x(\cdot, w)$ exhibits a jump in demand behavior at the price vector p, being x^n for all p^n but suddenly becoming the interval of consumption bundles $[\bar{x}, \bar{\bar{x}}]$ at p. It is upper hemicontinuous because $\bar{x}$ (the limiting optimum for p^n along the sequence) is an element of segment $[\bar{x}, \bar{\bar{x}}]$ (the set of optima at price vector p). See Section M.H of the Mathematical Appendix for further details on upper hemicontinuity.

Proposition 3.AA.1: Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preferences $\succsim$ on the consumption set $X = \mathbb{R}^L_+$. Then the derived demand correspondence $x(p, w)$ is upper hemicontinuous at all $(p, w) \gg 0$. Moreover, if $x(p, w)$ is a function [i.e., if $x(p, w)$ has a single element for all (p, w)], then it is continuous at all $(p, w) \gg 0$.

Proof: To verify upper hemicontinuity, suppose that we had a sequence $\{(p^n, w^n)\}_{n=1}^{\infty} \to (\bar{p}, \bar{w}) \gg 0$ and a sequence $\{x^n\}_{n=1}^{\infty}$ with $x^n \in x(p^n, w^n)$ for all n, such that $x^n \to \bar{x}$ and $\bar{x} \notin x(\bar{p}, \bar{w})$. Because $p^n \cdot x^n \leq w^n$ for all n, taking limits as $n \to \infty$, we conclude that $\bar{p} \cdot \bar{x} \leq \bar{w}$. Thus, $\bar{x}$ is a feasible consumption bundle when the budget set is $B_{\bar{p},\bar{w}}$. However, since it is not optimal in this set, it must be that $u(\tilde{x}) > u(\bar{x})$ for some $\tilde{x} \in B_{\bar{p},\bar{w}}$.

32. We use the notation $z^n \to z$ as synonymous with $z = \lim_{n \to \infty} z^n$. This definition of upper hemicontinuity applies only to correspondences that are "locally bounded" (see Section M.H of the Mathematical Appendix). Under our assumptions, the Walrasian demand correspondence satisfies this property at all $(p, w) \gg 0$.



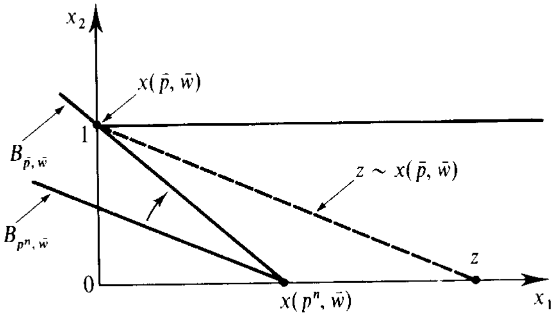


Figure 3.AA.2 (left)
Finding a bundle y such that $p \cdot y < w$ and $u(y) > u(\bar{x})$.

Figure 3.AA.3 (right)
The locally cheaper test fails at price wealth pair $(\bar{p}, \bar{w}) = (1, \bar{w}, \bar{w})$.

By the continuity of $u(\cdot)$, there is a y arbitrarily close to $\bar{x}$ such that $p \cdot y < w$ and $u(y) > u(\bar{x})$. This bundle y is illustrated in Figure 3.AA.2.

Note that if n is large enough, we will have $p^n \cdot y < w^n$ [since $(p^n, w^n) \to (p, w)$]. Hence, y is an element of the budget set B_{p^n, w^n}, and we must have $u(x^n) \geq u(y)$ because $x^n \in x(p^n, w^n)$. Taking limits as $n \to \infty$, the continuity of $u(\cdot)$ then implies that $u(\bar{x}) \geq u(y)$, which gives us a contradiction. We must therefore have $\bar{x} \in x(p, w)$, establishing upper hemicontinuity of $x(p, w)$.

The same argument also establishes continuity if $x(p, w)$ is in fact a function. ■

Suppose that the consumption set is an arbitrary closed set $X \subset \mathbb{R}^L_+$. Then the continuity (or upper hemicontinuity) property still follows at any $(\bar{p}, \bar{w})$ that passes the following (*locally cheaper consumption*) test: "Suppose that $x \in X$ is affordable (i.e., $\bar{p} \cdot x \leq \bar{w}$). Then there is a $y \in X$ arbitrarily close to x and that costs less than $\bar{w}$ (i.e., $\bar{p} \cdot y < \bar{w}$)." For example, in Figure 3.AA.3, commodity 2 is available only in indivisible unit amounts. The locally cheaper test then fails at the price wealth point $(\bar{p}, \bar{w}) = (1, \bar{w}, \bar{w})$, where a unit of good 2 becomes just affordable. You can easily verify by examining the figure [in which the dashed line indicates indifference between the points (0, 1) and z] that demand will fail to be upper hemicontinuous when $p_2 = \bar{w}$. In particular, for price–wealth points $(p^n, \bar{w})$ such that $p_1^n = 1$ and $p_2^n > \bar{w}$, $x(p^n, \bar{w})$ involves only the consumption of good 1; whereas at $(\bar{p}, \bar{w}) = (1, \bar{w}, \bar{w})$, we have $x(\bar{p}, w) = (0, 1)$. Note that the proof of Proposition 3.AA.1 fails when the locally cheaper consumption condition does not hold because we cannot find a consumption bundle y with the properties described there.

Differentiability

Proposition 3.AA.1 has established that if $x(p, w)$ is a function, then it is continuous. Often it is convenient that it be differentiable as well. We now discuss when this is so. We assume for the remaining paragraphs that $u(\cdot)$ is strictly quasiconcave and twice continuously differentiable and that $\nabla u(x) \neq 0$ for all x.

As we have shown in Section 3.D, the first-order conditions for the UMP imply that $x(p, w) \gg 0$ is, for some $\lambda > 0$, the unique solution of the system of $L + 1$ equations in $L + 1$ unknowns:

$$\nabla u(x) - \lambda p = 0$$
$$p \cdot x - w = 0.$$

Therefore, the *implicit function theorem* (see Section M.E of the Mathematical Appendix) tells us that the differentiability of the solution $x(p, w)$ as a function of the parameters (p, w) of the system depends on the Jacobian matrix of this system having a nonzero determinant. The Jacobian matrix [i.e., the derivative matrix of the $L + 1$ component functions with respect to the $L + 1$ variables (x, λ)] is

$$\begin{bmatrix} D^2u(x) & -p \\ p^{\mathrm{T}} & 0 \end{bmatrix}.$$

Since $\nabla u(x) = \lambda p$ and $\lambda > 0$, the determinant of this matrix is nonzero if and only if the determinant of the *bordered Hessian* of $u(x)$ at x is nonzero:

$$\begin{vmatrix} D^2u(x) & \nabla u(x) \\ [\nabla u(x)]^{\mathrm{T}} & 0 \end{vmatrix} \neq 0.$$

This condition has a straightforward geometric interpretation. It means that the indifference set through x has a nonzero curvature at x; it is not (even infinitesimally) flat. This condition is a slight technical strengthening of strict quasiconcavity [just as the strictly concave function $f(x) = -(x^4)$ has $f''(0) = 0$, a strictly quasiconcave function could have a bordered Hessian determinant that is zero at a point].

We conclude, therefore, that $x(p, w)$ is differentiable *if and only if* the determinant of the bordered Hessian of $u(\cdot)$ is nonzero at $x(p, w)$. It is worth noting the following interesting fact (which we shall not prove here): If $x(p, w)$ is differentiable at (p, w), then the Slutsky matrix $S(p, w)$ has maximal possible rank; that is, the rank of $S(p, w)$ equals $L - 1$.[33]

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33. This statement applies only to demand generated from a twice continuously differentiable utility function. It need not be true when this condition is not met. For example, the demand function $x(p, w) = (w/(p_1 + p_2), w/(p_1 + p_2))$ is differentiable, and it is generated by the utility function $u(x) = \mathrm{Min}\,\{x_1, x_2\}$, which is not twice continuously differentiable at all x. The substitution matrix for this demand function has all its entries equal to zero and therefore has rank equal to zero.

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EXERCISES

3.B.1^{A} In text.

3.B.2^{B} The preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ is said to be *weakly monotone* if and only if $x \geq y$ implies that $x \succsim y$. Show that if $\succsim$ is transitive, locally nonsatiated, and weakly monotone, then it is monotone.

3.B.3^{A} Draw a convex preference relation that is locally nonsatiated but is not monotone.

3.C.1^{B} Verify that the lexicographic ordering is complete, transitive, strongly monotone, and strictly convex.

3.C.2^{B} Show that if $u(\cdot)$ is a continuous utility function representing $\succsim$, then $\succsim$ is continuous.

3.C.3^{C} Show that if for every x the upper and lower contour sets $\{y \in \mathbb{R}^L_+ : y \succsim x\}$ and $\{y \in \mathbb{R}^L_+ : x \succsim y\}$ are closed, then $\succsim$ is continuous according to Definition 3.C.1.

3.C.4^{B} Exhibit an example of a preference relation that is not continuous but is representable by a utility function.

3.C.5^{C} Establish the following two results:

(a) A continuous $\succsim$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one; i.e., $u(\alpha x) = \alpha u(x)$ for all $\alpha > 0$.

(b) A continuous $\succsim$ on $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$ is quasilinear with respect to the first commodity if and only if it admits a utility function $u(x)$ of the form $u(x) = x_1 + \phi(x_2, \ldots, x_L)$. [*Hint*: The existence of some continuous utility representation is guaranteed by Proposition 3.G.1.]

After answering (a) and (b), argue that these properties of $u(\cdot)$ are cardinal.

3.C.6[B] Suppose that in a two-commodity world, the consumer's utility function takes the form $u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho}$. This utility function is known as the *constant elasticity of substitution* (or *CES*) utility function.

(a) Show that when $\rho = 1$, indifference curves become linear.

(b) Show that as $\rho \to 0$, this utility function comes to represent the same preferences as the (generalized) Cobb–Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.

(c) Show that as $\rho \to -\infty$, indifference curves become "right angles"; that is, this utility function has in the limit the indifference map of the Leontief utility function $u(x_1, x_2) = \text{Min}\{x_1, x_2\}$.

3.D.1[A] In text.

3.D.2[A] In text.

3.D.3[B] Suppose that $u(x)$ is differentiable and strictly quasiconcave and that the Walrasian demand function $x(p, w)$ is differentiable. Show the following:

(a) If $u(x)$ is homogeneous of degree one, then the Walrasian demand function $x(p, w)$ and the indirect utility function $v(p, w)$ are homogeneous of degree one [and hence can be written in the form $x(p, w) = w\tilde{x}(p)$ and $v(p, w) = w\tilde{v}(p)$] and the wealth expansion path (see Section 2.E) is a straight line through the origin. What does this imply about the wealth elasticities of demand?

(b) If $u(x)$ is strictly quasiconcave and $v(p, w)$ is homogeneous of degee one in w, then $u(x)$ must be homogeneous of degree one.

3.D.4[B] Let $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$ denote the consumption set, and assume that preferences are strictly convex and quasilinear. Normalize $p_1 = 1$.

(a) Show that the Walrasian demand functions for goods $2, \ldots, L$ are independent of wealth. What does this imply about the wealth effect (see Section 2.E) of demand for good 1?

(b) Argue that the indirect utility function can be written in the form $v(p, w) = w + \phi(p)$ for some function $\phi(\cdot)$.

(c) Suppose, for simplicity, that $L = 2$, and write the consumer's utility function as $u(x_1, x_2) = x_1 + \eta(x_2)$. Now, however, let the consumption set be $\mathbb{R}_+^2$ so that there is a nonnegativity constraint on consumption of the numeraire x_1. Fix prices p, and examine how the consumer's Walrasian demand changes as wealth w varies. When is the nonnegativity constraint on the numeraire irrelevant?

3.D.5[B] Consider again the CES utility function of Exercise 3.C.6, and assume that $\alpha_1 = \alpha_2 = 1$.

(a) Compute the Walrasian demand and indirect utility functions for this utility function.

(b) Verify that these two functions satisfy all the properties of Propositions 3.D.2 and 3.D.3.

(c) Derive the Walrasian demand correspondence and indirect utility function for the case of linear utility and the case of Leontief utility (see Exercise 3.C.6). Show that the CES Walrasian demand and indirect utility functions approach these as ρ approaches 1 and $-\infty$, respectively.

(d) The *elasticity of substitution between goods* 1 and 2 is defined as

$$\xi_{12}(p, w) = -\frac{\partial[x_1(p, w)/x_2(p, w)]}{\partial[p_1/p_2]} \frac{p_1/p_2}{x_1(p, w)/x_2(p, w)}.$$

Show that for the CES utility function, $\xi_{12}(p, w) = 1/(1 - \rho)$, thus justifying its name. What is $\xi_{12}(p, w)$ for the linear, Leontief, and Cobb–Douglas utility functions?

3.D.6[B] Consider the three-good setting in which the consumer has utility function $u(x) = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma$.

(a) Why can you assume that $\alpha + \beta + \gamma = 1$ without loss of generality? Do so for the rest of the problem.

(b) Write down the first-order conditions for the UMP, and derive the consumer's Walrasian demand and indirect utility functions. This system of demands is known as the *linear expenditure system* and is due to Stone (1954).

(c) Verify that these demand functions satisfy the properties listed in Propositions 3.D.2 and 3.D.3.

3.D.7[B] There are two commodities. We are given two budget sets B_{p^0,w^0} and B_{p^1,w^1} described, respectively, by $p^0 = (1, 1)$, $w^0 = 8$ and $p^1 = (1, 4)$, $w^1 = 26$. The observed choice at (p^0, w^0) is $x^0 = (4, 4)$. At (p^1, w^1), we have a choice x^1 such that $p \cdot x^1 = w^1$.

(a) Determine the region of permissible choices x^1 if the choices x^0 and x^1 are consistent with maximization of preferences.

(b) Determine the region of permissible choices x^1 if the choices x^0 and x^1 are consistent with maximization of preferences that are quasilinear with respect to the *first* good.

(c) Determine the region of permissible choices x^1 if the choices x^0 and x^1 are consistent with maximization of preferences that are quasilinear with respect to the *second* good.

(d) Determine the region of permissible choices x^1 if the choices x^0 and x^1 are consistent with maximization of preferences for which both goods are normal.

(e) Determine the region of permissible choices x^1 if the choices x^0 and x^1 are consistent with maximization of homothetic preferences.

[*Hint*: The ideal way to answer this exercise relies on (good) pictures as much as possible.]

3.D.8[A] Show that for all (p, w), $w\, \partial v(p, w)/\partial w = -p \cdot \nabla_p v(p, w)$.

3.E.1[A] In text.

3.E.2[A] In text.

3.E.3[B] Prove that a solution to the EMP exists if $p \gg 0$ and there is some $x \in \mathbb{R}^L_+$ satisfying $u(x) \geq u$.

3.E.4[B] Show that if the consumer's preferences $\succsim$ are convex, then $h(p, u)$ is a convex set. Also show that if $u(x)$ is strictly convex, then $h(p, u)$ is single-valued.

3.E.5[B] Show that if $u(\cdot)$ is homogeneous of degree one, then $h(p, u)$ and $e(p, u)$ are homogeneous of degree one in u [i.e., they can be written as $h(p, u) = \tilde{h}(p)u$ and $e(p, u) = \tilde{e}(p)u$].

3.E.6[B] Consider the constant elasticity of substitution utility function studied in Exercises 3.C.6 and 3.D.5 with $\alpha_1 = \alpha_2 = 1$. Derive its Hicksian demand function and expenditure function. Verify the properties of Propositions 3.E.2 and 3.E.3.

3.E.7[B] Show that if $\succsim$ is quasilinear with respect to good 1, the Hicksian demand functions for goods $2, \ldots, L$ do not depend on u. What is the form of the expenditure function in this case?

3.E.8[A] For the Cobb Douglas utility function, verify that the relationships in (3.E.1) and (3.E.4) hold. Note that the expenditure function can be derived by simply inverting the indirect utility function, and vice versa.

3.E.9[B] Use the relations in (3.E.1) to show that the properties of the indirect utility function identified in Proposition 3.D.3 imply Proposition 3.E.2. Likewise, use the relations in (3.E.1) to prove that Proposition 3.E.2 implies Proposition 3.D.3.

3.E.10[B] Use the relations in (3.E.1) and (3.E.4) and the properties of the indirect utility and expenditure functions to show that Proposition 3.D.2 implies Proposition 3.E.4. Then use these facts to prove that Proposition 3.E.3 implies Proposition 3.D.2.

3.F.1[B] Prove formally that a closed, convex set $K \subset \mathbb{R}^L$ equals the intersection of the half-spaces that contain it (use the separating hyperplane theorem).

3.F.2[A] Show by means of a graphic example that the separating hyperplane theorem does not hold for nonconvex sets. Then argue that if K is closed and not convex, there is always some $x \notin K$ that cannot be separated from K.

3.G.1[B] Prove that Proposition 3.G.1 is implied by Roy's identity (Proposition 3.G.4).

3.G.2[B] Verify for the case of a Cobb Douglas utility function that all of the propositions in Section 3.G hold.

3.G.3[B] Consider the (linear expenditure system) utility function given in Exercise 3.D.6.

(a) Derive the Hicksian demand and expenditure functions. Check the properties listed in Propositions 3.E.2 and 3.E.3.

(b) Show that the derivatives of the expenditure function are the Hicksian demand function you derived in (a).

(c) Verify that the Slutsky equation holds.

(d) Verify that the own-substitution terms are negative and that compensated cross-price effects are symmetric.

(e) Show that $S(p, w)$ is negative semidefinite and has rank 2.

3.G.4[B] A utility function $u(x)$ is *additively separable* if it has the form $u(x) = \sum_\ell u_\ell(x_\ell)$.

(a) Show that additive separability is a cardinal property that is preserved only under linear transformations of the utility function.

(b) Show that the induced ordering on any group of commodities is independent of whatever fixed values we attach to the remaining ones. It turns out that this ordinal property is not only necessary but also sufficient for the existence of an additive separable representation. [You should *not* attempt a proof. This is very hard. See Debreu (1960)].

(c) Show that the Walrasian and Hicksian demand functions generated by an additively separable utility function admit no inferior goods if the functions $u_\ell(\cdot)$ are strictly concave. (You can assume differentiability and interiority to answer this question.)

(d) (Harder) Suppose that all $u_\ell(\cdot)$ are identical and twice differentiable. Let $\hat{u}(\cdot) = u_\ell(\cdot)$. Show that if $-[t\hat{u}''(t)/\hat{u}'(t)] < 1$ for all t, then the Walrasian demand $x(p, w)$ has the so-called *gross substitute property*, i.e., $\partial x_\ell(p, w)/\partial p_k > 0$ for all ℓ and $k \neq \ell$.

3.G.5[C] (*Hicksian composite commodities.*) Suppose there are two groups of desirable commodities, x and y, with corresponding prices p and q. The consumer's utility function is $u(x, y)$, and her wealth is $w > 0$. Suppose that prices for goods y always vary in proportion to one another, so that we can write $q = \alpha q_0$. For any number $z \geq 0$, define the function

$$\tilde{u}(x, z) = \underset{y}{\text{Max}} \quad u(x, y)$$
$$\text{s.t. } q_0 \cdot y \leq z.$$

(a) Show that if we imagine that the goods in the economy are x and a single composite commodity z, that $\tilde{u}(x, z)$ is the consumer's utility function, and that α is the price of the composite commodity, then the solution to $\text{Max}_{x,z}\ \tilde{u}(x, z)$ s.t. $p \cdot x + \alpha z \le w$ will give the consumer's actual levels of x and $z = q_0 \cdot y$.

(b) Show that properties of Walrasian demand functions identified in Propositions 3.D.2 and 3.G.4 hold for $x(p, \alpha, w)$ and $z(p, \alpha, w)$.

(c) Show that the properties in Propositions 3.E.3, and 3.G.1 to 3.G.3 hold for the Hicksian demand functions derived using $\tilde{u}(x, z)$.

3.G.6[B] (F. M. Fisher) A consumer in a three-good economy (goods denoted x_1, x_2, and x_3; prices denoted p_1, p_2, p_3) with wealth level $w > 0$ has demand functions for commodities 1 and 2 given by

$$x_1 = 100 - 5\frac{p_1}{p_3} + \beta\frac{p_2}{p_3} + \delta\frac{w}{p_3}$$

$$x_2 = \alpha + \beta\frac{p_1}{p_3} + \gamma\frac{p_2}{p_3} + \delta\frac{w}{p_3}$$

where Greek letters are nonzero constants.

(a) Indicate how to calculate the demand for good 3 (but do not actually do it).

(b) Are the demand functions for x_1 and x_2 appropriately homogeneous?

(c) Calculate the restrictions on the numerical values of α, β, γ and δ implied by utility maximization.

(d) Given your results in part (c), for a fixed level of x_3 draw the consumer's indifference curve in the x_1, x_2 plane.

(e) What does your answer to (d) imply about the form of the consumer's utility function $u(x_1, x_2, x_3)$?

3.G.7[A] A striking duality is obtained by using the concept of *indirect demand function.* Fix w at some level, say $w = 1$; from now on, we write $x(p, 1) = x(p)$, $v(p, 1) = v(p)$. The *indirect demand function* $g(x)$ is the inverse of $x(p)$; that is, it is the rule that assigns to every commodity bundle $x \gg 0$ the price vector $g(x)$ such that $x = x(g(x), 1)$. Show that

$$g(x) = \frac{1}{x \cdot \nabla u(x)} \nabla u(x).$$

Deduce from Proposition 3.G.4 that

$$x(p) = \frac{1}{p \cdot \nabla v(p)} \nabla v(p).$$

Note that this is a completely symmetric expression. Thus, direct (Walrasian) demand is the normalized derivative of indirect utility, and indirect demand is the normalized derivative of direct utility.

3.G.8[B] The indirect utility function $v(p, w)$ is logarithmically homogeneous if $v(p, \alpha w) = v(p, w) + \ln \alpha$ for $\alpha > 0$ [in other words, $v(p, w) = \ln(v^*(p, w))$, where $v^*(p, w)$ is homogeneous of degree one]. Show that if $v(\cdot, \cdot)$ is logarithmically homogeneous, then $x(p, 1) = -\nabla_p v(p, 1)$.

3.G.9[C] Compute the Slutsky matrix from the indirect utility function.

3.G.10[B] For a function of the Gorman form $v(p, w) = a(p) + b(p)w$, which properties will the functions $a(\cdot)$ and $b(\cdot)$ have to satisfy for $v(p, w)$ to qualify as an indirect utility function?

3.G.11[B] Verify that an indirect utility function in Gorman form exhibits linear wealth-expansion curves.

3.G.12[B] What restrictions on the Gorman form correspond to the cases of homothetic and quasilinear preferences?

3.G.13[C] Suppose that the indirect utility function $v(p, w)$ is a polynomial of degree n on w (with coefficients that may depend on p). Show that any individual wealth-expansion path is contained in a linear subspace of at most dimension $n + 1$. Interpret.

3.G.14[A] The matrix below records the (Walrasian) demand substitution effects for a consumer endowed with rational preferences and consuming three goods at the prices $p_1 = 1$, $p_2 = 2$, and $p_3 = 6$:

$$\begin{bmatrix} -10 & ? & ? \\ ? & -4 & ? \\ 3 & ? & ? \end{bmatrix}.$$

Supply the missing numbers. Does the resulting matrix possess all the properties of a substitution matrix?

3.G.15[B] Consider the utility function

$$u = 2x_1^{1/2} + 4x_2^{1/2}.$$

(a) Find the demand functions for goods 1 and 2 as they depend on prices and wealth.

(b) Find the compensated demand function $h(\cdot)$.

(c) Find the expenditure function, and verify that $h(p, u) = \nabla_p e(p, u)$.

(d) Find the indirect utility function, and verify Roy's identity.

3.G.16[C] Consider the expenditure function

$$e(p, u) = \exp\left\{\sum_\ell \alpha_\ell \log p_\ell + \left(\prod_\ell p_\ell^{\beta_\ell}\right)u\right\}.$$

(a) What restrictions on $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are necessary for this to be derivable from utility maximization?

(b) Find the indirect utility that corresponds to it.

(c) Verify Roy's identity and the Slutsky equation.

3.G.17[B] [From Hausman (1981)] Suppose $L = 2$. Consider a "local" indirect utility function defined in some neighborhood of price–wealth pair $(\bar{p}, \bar{w})$ by

$$v(p, w) = -\exp(-bp_1/p_2)\left[\frac{w}{p_2} + \frac{1}{b}\left(a\frac{p_1}{p_2} + \frac{a}{b} + c\right)\right].$$

(a) Verify that the local demand function for the first good is

$$x_1(p, w) = a\frac{p_1}{p_2} + b\frac{w}{p_2} + c.$$

(b) Verify that the local expenditure function is

$$e(p, u) = -p_2 u \exp(bp_1/p_2) - \frac{1}{b}\left(ap_1 + \frac{a}{b}p_2 + cp_2\right).$$

(c) Verify that the local Hicksian demand function for the first commodity is

$$h_1(p, u) = -ub \exp(bp_1/p_2) - \frac{a}{b}.$$

3.G.18[C] Show that every good is related to every other good by a chain of (weak) substitutes; that is, for any goods ℓ and k, either $\partial h_\ell(p, u)/\partial p_k \geq 0$, or there exists a good r such that $\partial h_\ell(p, u)/\partial p_r \geq 0$ and $\partial h_r(p, u)/\partial p_k \geq 0$, or there is . . . , and so on. [*Hint*: Argue first the case of two commodities. Use next the insights on composite commodities gained in Exercise 3.G.5 to handle the case of three, and then L, commodities.]

3.H.1[C] Show that if $e(p, u)$ is continuous, increasing in u, homogeneous of degree one, nondecreasing, and concave in p, then the utility function $u(x) = \text{Sup}\,\{u: x \in V_u\}$, where $V_u = \{y: p \cdot y > e(p, u) \text{ for all } p \gg 0\}$, defined for $x \gg 0$, satisfies $e(p, u) = \text{Min}\,\{p \cdot x: u(x) \geq u\}$ for any $p \gg 0$.

3.H.2[B] Use Proposition 3.F.1 to argue that if $e(p, u)$ is differentiable in p, then there are no (strongly monotone) nonconvex preferences generating $e(\cdot)$.

3.H.3[A] How would you recover $v(p, w)$ from $e(p, u)$?

3.H.4[B] Suppose that we are given as primitive, not the Walrasian demand but the indirect demand function $g(x)$ introduced in Exercise 3.G.7. How would you go about recovering $\succsim$? Restrict yourself to the case $L = 2$.

3.H.5[B] Suppose you know the indirect utility function. How would you recover from it the expenditure function and the direct utility function?

3.H.6[B] Suppose that you observe the Walrasian demand functions $x_\ell(p, w) = \alpha_\ell w/p_\ell$ for all $\ell = 1, \ldots, L$ with $\sum_\ell \alpha_\ell = 1$. Derive the expenditure function of this demand system. What is the consumer's utility function?

3.H.7[B] Answer the following questions with reference to the demand function in Exercise 2.F.17.

(a) Let the utility associated with consumption bundle $x = (1, 1, \ldots, 1)$ be 1. What is the expenditure function $e(p, 1)$ associated with utility level $u = 1$? [*Hint*: Use the answer to (d) in Exercise 2.F.17.]

(b) What is the upper contour set of consumption bundle $x = (1, 1, \ldots, 1)$?

3.I.1[B] In text.

3.I.2[B] In text.

3.I.3[B] Consider a price change from initial price vector p^0 to new price vector $p^1 \leq p^0$ in which only the price of good ℓ changes. Show that $CV(p^0, p^1, w) > EV(p^0, p^1, w)$ if good ℓ is inferior.

3.I.4[B] Construct an example in which a comparison of $CV(p^0, p^1, w)$ and $CV(p^0, p^2, w)$ does not give the correct welfare ranking of p^1 versus p^2.

3.I.5[B] Show that if $u(x)$ is quasilinear with respect to the first good (and we fix $p_1 = 1$), then $CV(p^0, p^1, w) = EV(p^0, p^1, w)$ for any (p^0, p^1, w).

3.I.6[A] Suppose there are $i = 1, \ldots, I$ consumers with utility functions $u_i(x)$ and wealth w_i. We consider a change from p^0 to p^1. Show that if $\sum_i CV_i(p^0, p^1, w_i) > 0$ then we can find $\{w_i'\}_{i=1}^I$ such that $\sum_i w_i' \leq \sum_i w_i$ and $v_i(p^1, w_i') \geq v_i(p^0, w_i)$ for all i. That is, it is in principle possible to compensate everybody for the change in prices.

3.I.7[B] There are three commodities (i.e., $L = 3$), of which the third is a numeraire (let $p_3 = 1$). The market demand function $x(p, w)$ has

$$x_1(p, w) = a + bp_1 + cp_2$$
$$x_2(p, w) = d + ep_1 + gp_2.$$

(a) Give the parameter restrictions implied by utility maximization.

(b) Estimate the equivalent variation for a change of prices from $(p_1, p_2) = (1, 1)$ to $(\bar{p}_1, \bar{p}_2) = (2, 2)$. Verify that without appropriate symmetry, there is no path independence. Assume symmetry for the rest of the exercise.

(c) Let EV_1, EV_2, and EV be the equivalent variations for a change of prices from $(p_1, p_2) = (1, 1)$ to, respectively, $(2, 1)$, $(1, 2)$, and $(2, 2)$. Compare EV with $EV_1 + EV_2$ as a function of the parameters of the problem. Interpret.

(d) Suppose that the price increases in (c) are due to taxes. Denote the deadweight losses for each of the three experiments by DW_1, DW_2, and DW. Compare DW with $DW_1 + DW_2$ as a function of the parameters of the problem.

(e) Suppose the initial tax situation has prices $(p_1, p_2) = (1, 1)$. The government wants to raise a fixed (small) amount of revenue R through commodity taxes. Call t_1 and t_2 the tax rates for the two commodities. Determine the optimal tax rates as a function of the parameters of demand if the optimality criterion is the minimization of deadweight loss.

3.I.8[B] Suppose we are in a three-commodity market (i.e. $L = 3$). Letting $p_3 = 1$, the demand functions for goods 1 and 2 are

$$x_1(p, w) = a_1 + b_1 p_1 + c_1 p_2 + d_1 p_1 p_2$$
$$x_2(p, w) = a_2 + b_2 p_1 + c_2 p_2 + d_2 p_1 p_2.$$

(a) Note that the demand for goods 1 and 2 does not depend on wealth. Write down the most general class of utility functions whose demand has this property.

(b) Argue that if the demand functions in (a) are generated from utility maximization, then the values of the parameters cannot be arbitrary. Write down as exhaustive a list as you can of the restrictions implied by utility maximization. Justify your answer.

(c) Suppose that the conditions in (b) hold. The initial price situation is $p = (p_1, p_2)$, and we consider a change to $p' = (p'_1, p'_2)$. Derive a measure of welfare change generated in going from p to p'.

(d) Let the values of the parameters be $a_1 = a_2 = 3/2$, $b_1 = c_2 = 1$, $c_1 = b_2 = 1/2$, and $d_1 = d_2 = 0$. Suppose the initial price situation is $p = (1, 1)$. Compute the equivalent variation for a move to p' for each of the following three cases: (i) $p' = (2, 1)$, (ii) $p' = (1, 2)$, and (iii) $p' = (2, 2)$. Denote the respective answers by EV_1, EV_2, EV_3. Under which condition will you have $EV_3 = EV_1 + EV_2$? Discuss.

3.I.9[B] In a one-consumer economy, the government is considering putting a tax of t per unit on good ℓ and rebating the proceeds to the consumer (who nonetheless does not consider the effect of her purchases on the size of the rebate). Suppose that $s_{\ell\ell}(p, w) < 0$ for all (p, w). Show that the optimal tax (in the sense of maximizing the consumer's utility) is zero.

3.I.10[B] Construct an example in which the area variation measure approach incorrectly ranks p^0 and p^1. [*Hint*: Let the change from p^0 to p^1 involve a change in the price of more than one good.]

3.I.11[B] Suppose that we know not only p^0, p^1, and x^0 but also $x^1 = x(p^1, w)$. Show that if $(p^1 - p^0) \cdot x^1 > 0$, then the consumer must be worse off at price–wealth situation (p^1, w) than at (p^0, w). Interpret this test as a first-order approximation to the expenditure function at p^1.

Also show that an alternative way to write this test is $p^0 \cdot (x^1 - x^0) < 0$, and depict the test for the case where $L = 2$ in (x_1, x_2) space. [*Hint*: Locate the point x^0 on the set $\{x \in \mathbb{R}^L_+ : u(x) = u^0\}$.]

3.I.12[B] Extend the compensating and equivalent variation measures of welfare change to the case of changes in both prices and wealth, so that we change from (p^0, w^0) to (p^1, w^1). Also extend the "partial information" test developed in Section 3.I to this case.

3.J.1[C] Show that when $L = 2$, $x(p, w)$ satisfies the strong axiom if and only if it satisfies the weak axiom.

3.AA.1[B] Suppose that the consumption set is $X = \{x \in \mathbb{R}^2_+ : x_1 + x_2 \geq 1\}$ and the utility function is $u(x) = x_2$. Represent graphically, and show (a) that the locally cheaper consumption test fails at $(p, w) = (1, 1, 1)$ and (b) that market demand is not continuous at this point. Interpet economically.

3.AA.2[C] Under the conditions of Proposition 3.AA.1, show that $h(p, u)$ is upper hemicontinuous and that $e(p, u)$ is continuous (even if we replace minimum by infimum and allow $p \geq 0$). Also, assuming that $h(p, u)$ is a function, give conditions for its differentiability.