4 Aggregate Demand

4.A Introduction

For most questions in economics, the aggregate behavior of consumers is more important than the behavior of any single consumer. In this chapter, we investigate the extent to which the theory presented in Chapters 1 to 3 can be applied to *aggregate demand*, a suitably defined sum of the demands arising from all the economy's consumers. There are, in fact, a number of different properties of individual demand that we might hope would also hold in the aggregate. Which ones we are interested in at any given moment depend on the particular application at hand.

In this chapter, we ask three questions about aggregate demand:

(i) Individual demand can be expressed as a function of prices and the individual's wealth level. *When can aggregate demand be expressed as a function of prices and aggregate wealth?*

(ii) Individual demand derived from rational preferences necessarily satisfies the weak axiom of revealed preference. *When does aggregate demand satisfy the weak axiom?* More generally, when can we apply in the aggregate the demand theory developed in Chapter 2 (especially Section 2.F)?

(iii) Individual demand has welfare significance; from it, we can derive measures of welfare change for the consumer, as discussed in Section 3.I. *When does aggregate demand have welfare significance?* In particular, when do the welfare measures discussed in Section 3.I have meaning when they are computed from the aggregate demand function?

These three questions could, with a grain of salt, be called the *aggregation theories of*, respectively, *the econometrician*, *the positive theorist*, and *the welfare theorist*.

The econometrician is interested in the degree to which he can impose a simple structure on aggregate demand functions in estimation procedures. One aspect of these concerns, which we address here, is the extent to which aggregate demand can be accurately modeled as a function of only *aggregate* variables, such as aggregate (or, equivalently, average) consumer wealth. This question is important because the econometrician's data may be available only in an aggregate form.

The positive (behavioral) theorist, on the other hand, is interested in the degree to which the positive restrictions of individual demand theory apply in the aggregate. This can be significant for deriving predictions from models of market equilibrium in which aggregate demand plays a central role.[1]

The welfare theorist is interested in the normative implications of aggregate demand. He wants to use the measures of welfare change derived in Section 3.I to evaluate the welfare significance of changes in the economic environment. Ideally, he would like to treat aggregate demand as if it were generated by a "representative consumer" and use the changes in this fictional individual's welfare as a measure of aggregate welfare.

Although the conditions we identify as important for each of these aggregation questions are closely related, the questions being asked in the three cases are conceptually quite distinct. Overall, we shall see that, in all three cases, very strong restrictions will need to hold for the desired aggregation properties to obtain. We discuss these three questions, in turn, in Sections 4.B to 4.D.

Finally, Appendix A discusses the regularizing (i.e., "smoothing") effects arising from aggregation over a large number of consumers.

4.B Aggregate Demand and Aggregate Wealth

Suppose that there are I consumers with rational preference relations $\succsim_i$ and corresponding Walrasian demand functions $x_i(p, w_i)$. In general, given prices $p \in \mathbb{R}^L$ and wealth levels $(w_1, \dots, w_I)$ for the I consumers, aggregate demand can be written as

$$x(p, w_1, \dots, w_I) = \sum_{i=1}^{I} x_i(p, w_i).$$

Thus, aggregate demand depends not only on prices but also on the specific wealth levels of the various consumers. In this section, we ask when we are justified in writing aggregate demand in the simpler form $x(p, \sum_i w_i)$, where aggregate demand depends only on aggregate wealth $\sum_i w_i$.

For this property to hold in all generality, aggregate demand must be identical for any two distributions of the same total amount of wealth across consumers. That is, for any $(w_1, \dots, w_I)$ and $(w'_1, \dots, w'_I)$ such that $\sum_i w_i = \sum_i w'_i$, we must have $\sum_i x_i(p, w_i) = \sum_i x_i(p, w'_i)$.

To examine when this condition is satisfied, consider, starting from some initial distribution $(w_1, \dots, w_I)$, a differential change in wealth $(dw_1, \dots, dw_I) \in \mathbb{R}^I$ satisfying $\sum_i dw_i = 0$. If aggregate demand can be written as a function of aggregate wealth, then assuming differentiability of the demand functions, we must have

$$\sum_i \frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} dw_i = 0 \quad \text{for every } \ell.$$

This can be true for all redistributions $(dw_1, \dots, dw_I)$ satisfying $\sum_i dw_i = 0$ and from any initial wealth distribution $(w_1, \dots, w_I)$ if and only if the coefficients of the different

1. The econometrician may also be interested in these questions because a priori restrictions on the properties of aggregate demand can be incorporated into his estimation procedures.

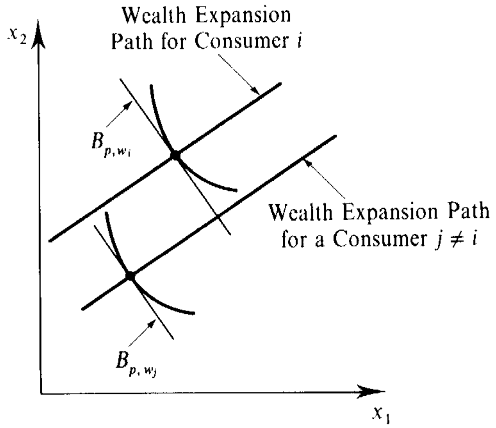


Figure 4.B.1
Invariance of aggregate demand to redistribution of wealth implies wealth expansion paths that are straight and parallel across consumers.

dw_i are equal; that is,

$$\frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} = \frac{\partial x_{\ell j}(p, w_j)}{\partial w_j} \tag{4.B.1}$$

for every ℓ, any two individuals i and j, and all $(w_1, \ldots, w_I)$.[2]

In short, for any fixed price vector p, and any commodity ℓ, the wealth effect at p must be the same whatever consumer we look at and whatever his level of wealth.[3] It is indeed fairly intuitive that in this case, the individual demand changes arising from any wealth redistribution across consumers will cancel out. Geometrically, the condition is equivalent to the statement that all consumers' wealth expansion paths are parallel, straight lines. Figure 4.B.1 depicts parallel, straight wealth expansion paths.

One special case in which this property holds arises when all consumers have identical preferences that are homothetic. Another is when all consumers have preferences that are quasilinear with respect to the same good. Both cases are examples of a more general result shown in Proposition 4.B.1.

Proposition 4.B.1: A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that preferences admit indirect utility functions of the Gorman form with the coefficients on w_i the same for every consumer i. That is:

$$v_i(p, w_i) = a_i(p) + b(p)w_i.$$

Proof: You are asked to establish sufficiency in Exercise 4.B.1 (this is not too difficult; use Roy's identity). Keep in mind that we are neglecting boundaries (alternatively, the significance of a result such as this is only local). You should not attempt to prove necessity. For a discussion of this result, see Deaton and Muellbauer (1980). ■

2. As usual, we are neglecting boundary constraints; hence, strictly speaking, the validity of our claims in this section is only local.

3. Note that $\partial x_{\ell i}(p, w_i)/\partial w_i = \partial x_{\ell i}(p, w_i')/\partial w_i$ for all $w_i \neq w_i'$ because for any values of $w_j, j \neq i$, (4.B.1) must hold for the wealth distributions $(w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_I)$ and $(w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_I)$. Hence, $\partial x_{\ell i}(p, w_i)/\partial w_i = \partial x_{\ell j}(p, w_j)/\partial w_j = \partial x_{\ell i}(p, w_i')/\partial w_i$ for any $j \neq i$.

Thus, aggregate demand can be written as a function of aggregate wealth if and only if all consumers have preferences that admit indirect utility functions of the Gorman form with equal wealth coefficients $b(p)$. Needless to say, this is a very restrictive condition on preferences.[4]

Given this conclusion, we might ask whether less restrictive conditions can be obtained if we consider aggregate demand functions that depend on a wider set of aggregate variables than just the total (or, equivalently, the mean) wealth level. For example, aggregate demand might be allowed to depend on both the mean and the variance of the statistical distribution of wealth or even on the whole statistical distribution itself. Note that the latter condition is still restrictive. It implies that aggregate demand depends only on how many rich and poor there are, not on who in particular is rich or poor.

These more general forms of dependence on the distribution of wealth are indeed valid under weaker conditions than those required for aggregate demand to depend only on aggregate wealth. For a trivial example, note that aggregate demand depends only on the statistical distribution of wealth whenever all consumers possess identical but otherwise arbitrary preferences and differ only in their wealth levels. We shall not pursue this topic further here; good references are Deaton and Muellbauer (1980), Lau (1982) and Jorgenson (1990).

There is another way in which we might be able to get a more positive answer to our question. So far, the test that we have applied is whether the aggregate demand function can be written as a function of aggregate wealth for *any* distribution of wealth across consumers. The requirement that this be true for every conceivable wealth distribution is a strong one. Indeed, in many situations, individual wealth levels may be generated by some underlying process that restricts the set of individual wealth levels which can arise. If so, it may still be possible to write aggregate demand as a function of prices and aggregate wealth.

For example, when we consider general equilibrium models in Part IV, individual wealth is generated by individuals' shareholdings of firms and by their ownership of given, fixed stocks of commodities. Thus, the individual levels of real wealth are determined as a function of the prevailing price vector.

Alternatively, individual wealth levels may be determined in part by various government programs that redistribute wealth across consumers (see Section 4.D). Again, these programs may limit the set of possible wealth distributions that may arise.

To see how this can help, consider an extreme case. Suppose that individual i's wealth level is generated by some process that can be described as a function of prices p and aggregate wealth w, $w_i(p, w)$. This was true, for example, in the general equilibrium illustration above. Similarly, the government program may base an individual's taxes (and hence his final wealth position) on his wage rate and the total (real) wealth of the society. We call a family of functions $(w_1(p, w), \ldots, w_I(p, w))$ with $\sum_i w_i(p, w) = w$ for all (p, w) a *wealth distribution rule*. When individual wealth levels

4. Recall, however, that it includes some interesting and important classes of preferences. For example, if preferences are quasilinear with respect to good ℓ, then there is an indirect utility of the form $a_i(p) + w_i/p_\ell$, which, letting $b(p) = 1/p_\ell$, we can see is of the Gorman type with identical $b(p)$.

are generated by a wealth distribution rule, we can indeed *always* write aggregate demand as a function $x(p, w) = \sum_i x_i(p, w_i(p, w))$, and so aggregate demand depends only on prices and aggregate wealth.

4.C Aggregate Demand and the Weak Axiom

To what extent do the positive properties of individual demand carry over to the aggregate demand function $x(p, w_1, \dots, w_I) = \sum_i x_i(p, w_i)$? We can note immediately three properties that do: continuity, homogeneity of degree zero, and Walras' law [that is, $p \cdot x(p, w_1, \dots, w_I) = \sum_I w_i$ for all $(p, w_1, \dots, w_I)$]. In this section, we focus on the conditions under which aggregate demand also satisfies the weak axiom, arguably the most central positive property of the individual Walrasian demand function.

To study this question, we would like to operate on an aggregate demand written in the form $x(p, w)$, where w is aggregate wealth. This is the form for which we gave the definition of the weak axiom in Chapter 2. We accomplish this by supposing that there is a wealth distribution rule $(w_1(p, w), \dots, w_I(p, w))$ determining individual wealths from the price vector and total wealth. We refer to the end of Section 4.B for a discussion of wealth distribution rules.[5] With the wealth distribution rule at our disposal, aggregate demand can automatically be written as

$$x(p, w) = \sum_i x_i(p, w_i(p, w)).$$

Formally, therefore, the aggregate demand function $x(p, w)$ depends then only on aggregate wealth and is therefore a market demand function in the sense discussed in Chapter 2.[6] We now investigate the fulfillment of the weak axiom by $x(\cdot, \cdot)$.

In point of fact, and merely for the sake of concreteness, we shall be even more specific and focus on a particularly simple example of a distribution rule. Namely, we restrict ourselves to the case in which relative wealths of the consumers remain fixed, that is, are independent of prices. Thus, we assume that we are given wealth shares $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$, so that $w_i(p, w) = \alpha_i w$ for every level $w \in \mathbb{R}$ of aggregate wealth.[7] We have then

$$x(p, w) = \sum_i x_i(p, \alpha_i w).$$

We begin by recalling from Chapter 2 the definition of the weak axiom.

Definition 4.C.1: The aggregate demand function $x(p, w)$ satisfies the weak axiom (WA) if $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$ imply $p' \cdot x(p, w) > w'$ for any (p, w) and (p', w').

5. There is also a methodological advantage to assuming the presence of a wealth distribution rule. It avoids confounding different aggregation issues because the aggregation problem studied in Section 4.B (invariance of demand to redistributions) is then entirely assumed away.

6. Note that it assigns commodity bundles to price wealth combinations, and, provided every $w_i(\cdot, \cdot)$ is continuous and homogeneous of degree one, that it is continuous, homogeneous of degree zero, and satisfies Walras's law.

7. Observe that this distribution rule amounts to leaving the wealth levels $(w_1, \dots, w_I)$ unaltered and considering only changes in the price vector p. This is because the homogeneity of degree zero of $x(p, w_1, \dots, w_I)$ implies that any proportional change in wealths can also be captured by a proportional change in prices. The description by means of shares is, however, analytically more convenient.

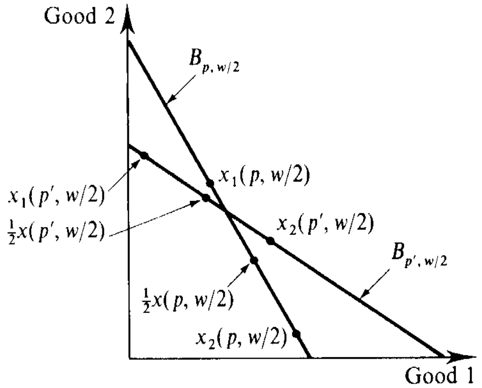


Figure 4.C.1
Failure of aggregate demand to satisfy the weak axiom.

We next provide an example illustrating that aggregate demand may not satisfy the weak axiom.

Example 4.C.1: *Failure of Aggregate Demand to Satisfy the WA.* Suppose that there are two commodities and two consumers. Wealth is distributed equally so that $w_1 = w_2 = w/2$, where w is aggregate wealth. Two price vectors p and p' with corresponding individual demands $x_1(p, w/2)$ and $x_2(p, w/2)$ under p, and $x_1(p', w/2)$ and $x_2(p', w/2)$ under p', are depicted in Figure 4.C.1.

These individual demands satisfy the weak axiom, but the aggregate demands do not. Figure 4.C.1 shows the vectors $\frac{1}{2}x(p, w)$ and $\frac{1}{2}x(p', w)$, which are equal to the average of the two consumers' demands; (and so for each price vector, they must lie at the midpoint of the line segment connecting the two individuals' consumption vectors). As illustrated in the figure, we have

$$\tfrac{1}{2}p \cdot x(p', w) < w/2 \qquad \text{and} \qquad \tfrac{1}{2}p' \cdot x(p, w) < w/2,$$

which (multiply both sides by 2) constitutes a violation of the weak axiom at the price-wealth pairs considered. ■

The reason for the failure illustrated in Example 4.C.1 can be traced to wealth effects. Recall from Chapter 2 (Proposition 2.F.1) that $x(p, w)$ satisfies the weak axiom if and only if it satisfies the law of demand for *compensated* price changes. Precisely, if and only if for any (p, w) and any price change p' that is compensated [so that $w' = p' \cdot x(p, w)$], we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0, \tag{4.C.1}$$

with strict inequality if $x(p, w) \neq x(p', w')$.[8]

If the price–wealth change under consideration, say from (p, w) to (p', w'), happened to be a compensated price change for *every* consumer i—that is, if $\alpha_i w' = p' \cdot x_i(p, \alpha_i w)$ for all i—then because individual demand satisfies the weak axiom, we would know (again by Proposition 2.F.1) that for all $i = 1, \ldots, I$:

$$(p' - p) \cdot [x_i(p', \alpha_i w') - x_i(p, \alpha_i w)] \leq 0, \tag{4.C.2}$$

8. Note that if $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$, then we must have $p' \cdot x(p, w) > w'$, in agreement with the weak axiom.

with strict inequality if $x_i(p', \alpha_i w) \neq x_i(p, \alpha_i w')$. Adding (4.C.2) over i gives us precisely (4.C.1). Thus, we conclude that aggregate demand must satisfy the WA for any price–wealth change that is compensated for every consumer.

The difficulty arises because a price–wealth change that is compensated in the aggregate, so that $w' = p' \cdot x(p, w)$, need not be compensated for each individual; we may well have $\alpha_i w' \neq p' \cdot x_i(p, \alpha_i w)$ for some or all i. If so, the individual wealth effects [which, except for the condition $p \cdot D_{w_i} x(p, \alpha_i w) = 1$, are essentially unrestricted] can play havoc with the well-behaved but possibly small individual substitution effects. The result may be that (4.C.2) fails to hold for some i, thus making possible the failure of the similar expression (4.C.1) in the aggregate.

Given that a property of individual demand as basic as the WA cannot be expected to hold generally for aggregate demand, we might wish to know whether there are any restrictions on individual preferences under which it must be satisfied. The preceding discussion suggests that it may be worth exploring the implications of assuming that the law of demand, expression (4.C.2), holds at the individual level for price changes that are left uncompensated. Suppose, indeed, that given an initial position (p, w_i), we consider a price change p' that is not compensated, namely, we leave $w'_i = w_i$. If (4.C.2) nonetheless holds, then by addition so does (4.C.1). More formally, we begin with a definition.

Definition 4.C.2: The individual demand function $x_i(p, w_i)$ satisfies the *uncompensated law of demand* (*ULD*) property if

$$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0 \qquad (4.C.3)$$

for any p, p', and w_i, with strict inequality if $x_i(p', w_i) \neq x_i(p, w_i)$. The analogous definition applies to the aggregate demand function $x(p, w)$.

In view of our discussion of the weak axiom in Section 2.F, the following differential version of the ULD property should come as no surprise (you are asked to prove it in Exercise 4.C.1):

> If $x_i(p, w_i)$ satisfies the ULD property, then $D_p x_i(p, w_i)$ is negative semidefinite; that is, $dp \cdot D_p x_i(p, w_i)\, dp \leq 0$ for all dp.

As with the weak axiom, there is a converse to this:

> If $D_p x_i(p, w_i)$ is negative definite for all p, then $x_i(p, w_i)$ satisfies the ULD property.

The analogous differential version holds for the aggregate demand function $x(p, w)$.

The great virtue of the ULD property is that, in contrast with the WA, it does, in fact, aggregate. Adding the individual condition (4.C.3) for $w_i = \alpha_i w$ gives us $(p' - p) \cdot [x(p', w) - x(p, w)] \leq 0$, with strict inequality if $x(p, w) \neq x'(p, w)$. This leads us to Proposition 4.C.1.

Proposition 4.C.1: If every consumer's Walrasian demand function $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property, so does the aggregate demand $x(p, w) = \sum_i x_i(p, \alpha_i w)$. As a consequence, the aggregate demand $x(p, w)$ satisfies the weak axiom.

Proof: Consider any (p, w), (p', w) with $x(p, w) \neq x(p', w)$. We must have

$$x_i(p, \alpha_i w) \neq x_i(p', \alpha_i w)$$

for some i. Therefore, adding (4.C.3) over i, we get

$$(p' - p)\cdot[x(p, w) - x(p', w)] < 0.$$

This holds for all p, p', and w.

To verify the WA, take any (p, w), (p', w') with $x(p, w) \neq x(p', w')$ and $p\cdot x(p', w') \leq w$.[9] Define $p'' = (w/w')p'$. By homogeneity of degree zero, we have $x(p'', w) = x(p', w')$. From $(p'' - p)\cdot[x(p'', w) - x(p, w)] < 0$, $p\cdot x(p'', w) \leq w$, and Walras' law, it follows that $p''\cdot x(p, w) > w$. That is, $p'\cdot x(p, w) > w'$. ∎

How restrictive is the ULD property as an axiom of individual behavior? It is clearly not implied by preference maximization (see Exercise 4.C.3). Propositions 4.C.2 and 4.C.3 provide sufficient conditions for individual demands to satisfy the ULD property.

Proposition 4.C.2: If $\succsim_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property.

Proof: We consider the differentiable case [i.e., we assume that $x_i(p, w_i)$ is differentiable and that $\succsim_i$ is representable by a differentiable utility function]. The matrix $D_p x_i(p, w_i)$ is

$$D_p x_i(p, w_i) = S_i(p, w_i) - \frac{1}{w_i} x_i(p, w_i)\, x_i(p, w_i)^T,$$

where $S_i(p, w_i)$ is consumer i's Slutsky matrix. Because $[dp\cdot x_i(p, w_i)]^2 > 0$ except when $dp\cdot x_i(p, w_i) = 0$ and $dp\cdot S_i(p, w_i)\, dp < 0$ except when dp is proportional to p, we can conclude that $D_p x_i(p, w_i)$ is negative definite, and so the ULD condition holds. ∎

In Proposition 4.C.2, the conclusion is obtained with minimal help from the substitution effects. Those could all be arbitrarily small. The wealth effects by themselves turn out to be sufficiently well behaved. Unfortunately, the homothetic case is the only one in which this is so (see Exercise 4.C.4). More generally, for the ULD property to hold, the substitution effects (which are always well behaved) must be large enough to overcome possible "perversities" coming from the wealth effects. The intriguing result in Proposition 4.3.C [due to Mitiushin and Polterovich (1978) and Milleron (1974); see Mas-Colell (1991) for an account and discussion of this result] gives a concrete expression to this relative dominance of the substitution effects.

Proposition 4.C.3: Suppose that $\succsim_i$ is defined on the consumption set $X = \mathbb{R}^L_+$ and is representable by a twice continuously differentiable concave function $u_i(\cdot)$. If

$$-\frac{x_i\cdot D^2 u_i(x_i) x_i}{x_i\cdot \nabla u_i(x_i)} < 4 \quad \text{for all } x_i,$$

then $x_i(p, w_i)$ satisfies the unrestricted law of demand (ULD) property.

9. Strictly speaking, this proof is required because although we know that the WA is equivalent to the law of demand for compensated price changes, we are now dealing with uncompensated price changes.

The proof of Proposition 4.C.3 will not be given. The courageous reader can attempt it in Exercise 4.C.5.

The condition in Proposition 4.C.3 is not an extremely stringent one. In particular, notice how amply the homothetic case fits into it (Exercise 4.C.6). So, to the question "How restrictive is the ULD property as an axiom of individual behavior?" perhaps we can answer: "restrictive, but not extremely so."[10]

Note, in addition, that for the ULD property to hold for aggregate demand, it is not necessary that the ULD be satisfied at the individual level. It may arise out of aggregation itself. The example in Proposition 4.C.4, due to Hildenbrand (1983), is not very realistic, but it is nonetheless highly suggestive.

Proposition 4.C.4: Suppose that all consumers have identical preferences $\succsim$ defined on $\mathbb{R}^L_+$ [with individual demand functions denoted $\tilde{x}(p, w)$] and that individual wealth is uniformly distributed on an interval $[0, \bar{w}]$ (strictly speaking, this requires a continuum of consumers). Then the aggregate (rigorously, the average) demand function

$$x(p) = \int_0^{\bar{w}} \tilde{x}(p, w)\, dw$$

satisfies the unrestricted law of demand (ULD) property.

Proof: Consider the differentiable case. Take $v \neq 0$. Then

$$v \cdot Dx(p)v = \int_0^{w} v \cdot D_p \tilde{x}(p, w) v \, dw.$$

Also

$$D_p \tilde{x}(p, w) = S(p, w) - D_w \tilde{x}(p, w) \tilde{x}(p, w)^T,$$

where $S(p, w)$ is the Slutsky matrix of the individual demand function $x(\cdot, \cdot)$ at (p, w). Hence,

$$v \cdot Dx(p)v = \int_0^{w} v \cdot S(p, w) v \, dw - \int_0^{w} (v \cdot D_w \tilde{x}(p, w))(v \cdot \tilde{x}(p, w)) \, dw.$$

The first term of this sum is negative, unless v is proportional to p. For the second, note that

$$2(v \cdot D_w \tilde{x}(p, w))(v \cdot \tilde{x}(p, w)) = \frac{d(v \cdot \tilde{x}(p, w))^2}{dw}.$$

So

$$-\int_0^{w} (v \cdot D_w \tilde{x}(p, w))(v \cdot \tilde{x}(p, w)) \, dw = -\frac{1}{2} \int_0^{\bar{w}} \frac{d(v \cdot \tilde{x}(p, w))^2}{dw} \, dw = -\frac{1}{2} (v \cdot \tilde{x}(p, \bar{w}))^2 \leq 0,$$

where we have used $\tilde{x}(p, 0) = 0$. Observe that the sign is negative when v is proportional to p. ■

Recall that the ULD property is additive across groups of consumers. Therefore, what we need in order to apply Proposition 4.C.4 is, not that preferences be identical, but that for every preference relation, the distribution of wealth conditional on that preference be uniform over

10. Not to misrepresent the import of this claim, we should emphasize that Proposition 4.C.1, which asserts that the ULD property is preserved under addition, holds for the price-independent distribution rules that we are considering in this section. When the distribution of real wealth may depend on prices (as it typically will in the general equilibrium applications of Part IV), then aggregate demand may violate the WA even if individual demand satisfies the ULD property (see Exercise 4.C.13). We discuss this point further in Section 17.F.

some interval that includes the level 0 (in fact, a nonincreasing density function is enough; see Exercise 4.C.7).

One lesson of Proposition 4.C.4 is that the properties of aggregate demand will depend on how preferences and wealth are distributed. We could therefore pose the problem quite generally and ask which distributional conditions on preferences and wealth will lead to satisfaction of the weak axiom by aggregate demand.[11]

As mentioned in Section 2.F, a market demand function $x(p, w)$ can be shown to satisfy the WA if for all (p, w), the Slutsky matrix $S(p, w)$ derived from the function $x(p, w)$ satisfies $dp \cdot S(p, w)\, dp < 0$ for every $dp \neq 0$ not proportional to p. We now examine when this property might hold for the aggregate demand function.

The Slutsky equation for the aggregate demand function is

$$S(p, w) = D_p x(p, w) + D_w x(p, w)\, x(p, w)^{\mathrm{T}}. \tag{4.C.4}$$

Or, since $x(p, w) = \sum_i x_i(p, \alpha_i w)$,

$$S(p, w) = D_p x(p, w) + [\textstyle\sum_i \alpha_i D_{w_i} x_i(p, \alpha_i w)]\, x(p, w)^{\mathrm{T}} \tag{4.C.5}$$

Next, let $S_i(p, w_i)$ denote the individual Slutsky matrices. Adding the individual Slutsky equations gives

$$\textstyle\sum_i S_i(p, \alpha_i w) = \sum_i D_p x_i(p, \alpha_i w) + \sum_i D_{w_i} x_i(p, \alpha_i w)\, x_i(p, \alpha_i w)^{\mathrm{T}} \tag{4.C.6}$$

Since $D_p x(p, w) = \sum_i D_p x_i(p, \alpha_i w)$, we can substitute (4.C.6) into (4.C.5) to get

$$S(p, w) = \textstyle\sum_i S_i(p, w_i) - \sum_i \alpha_i [D_{w_i} x_i(p, \alpha_i w) - D_w x(p, w)] \left[\frac{1}{\alpha_i} x_i(p, \alpha_i w) - x(p, w) \right]^{\mathrm{T}}. \tag{4.C.7}$$

Note that because of wealth effects, the Slutsky matrix of aggregate demand is *not* the sum of the individual Slutsky matrices. The difference

$$\begin{aligned} C(p, w) &= \textstyle\sum_i S_i(p, \alpha w) - S(p, w) \\ &= \textstyle\sum_i \alpha_i [D_{w_i} x_i(p, \alpha_i w) - D_w x(p, w)] \left[\frac{1}{\alpha_i} x_i(p, \alpha_i w) - x(p, w) \right]^{\mathrm{T}} \end{aligned} \tag{4.C.8}$$

is a covariance matrix between wealth effect vectors $D_{w_i} x_i(p, \alpha_i w)$ and proportionately adjusted consumption vectors $(1/\alpha_i) x_i(p, \alpha_i w)$. The former measures how the marginal dollar is spent across commodities; the latter measures the same thing for the average dollar [e.g., $(1/\alpha_i w) x_{\ell i}(p, \alpha_i w)$ is the per-unit-of-wealth consumption of good ℓ by consumer i]. Every "observation" receives weight α_i. Note also that, as it should be, we have

$$\textstyle\sum_i \alpha_i [D_{w_i} x_i(p, \alpha_i w) - D_w x(p, w)] = 0 \quad \text{and} \quad \sum_i \alpha_i [(1/\alpha_i) x_i(p, \alpha_i w) - x(p, w)] = 0.$$

For an individual Slutsky matrix $S_i(\cdot, \cdot)$ we always have $dp \cdot S_i(p, \alpha_i w)\, dp < 0$ for $dp \neq 0$ not proportional to p. Hence, a *sufficient* condition for the Slutsky matrix of aggregate demand to have the desired property is that $C(p, w)$ be positive semidefinite. Speaking loosely, this will be the case if, on average, there is a *positive association* across consumers between consumption (per unit of wealth) in one commodity and the wealth effect for that commodity.

Figure 4.C.2(a) depicts a case for $L = 2$ in which, assuming a uniform distribution of wealth across consumers, this association is positive: Consumers with higher-than-average

11. In the next few paragraphs, we follow Jerison (1982) and Freixas and Mas-Colell (1987).

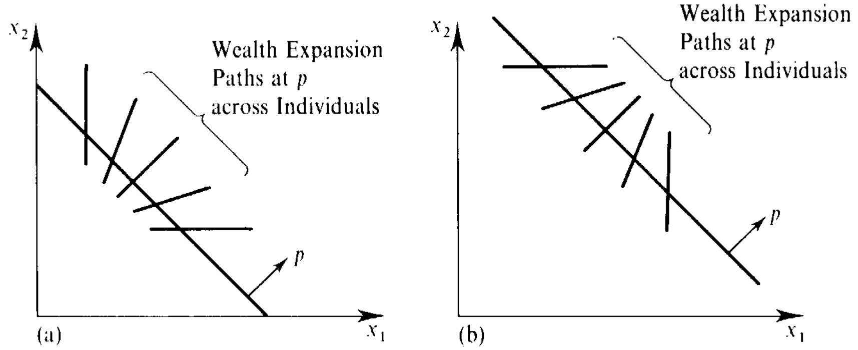


Figure 4.C.2
The relation across consumers between expenditure per unit of wealth on a commodity and its wealth effect when all consumers have the same wealth.
(a) Positive relation.
(b) Negative relation.

consumption of one good spend a higher-than-average fraction of their last unit of wealth on that good. The association is negative in Figure 4.C.2(b).[12,13]

From the preceding derivation, we can see that aggregate demand satisfies the WA in two cases of interest: (i) All the $D_{w_i}x_i(p, \alpha_i w)$ are equal (there are equal wealth effects), and (ii) all the $(1/\alpha_i)x_i(p, \alpha_i w)$ are equal (there is proportional consumption). In both cases, we have $C(p, w) = 0$, and so $dp \cdot S(p, w)\, dp < 0$ whenever $dp \neq 0$ is not proportional to p.

Case (i) has important implications. In particular, if every consumer has indirect utility functions of the Gorman form $v_i(p, w_i) = a_i(p) + b(p)w$, with the coefficient $b(p)$ identical across consumers, then (as we saw in Section 4.B) the wealth effects are the same for all consumers and we can therefore conclude that the WA is satisfied. We know from Section 4.B that one is led to this family of indirect utility functions by the requirement that aggregate demand be invariant to redistribution of wealth. Thus, aggregate demand satisfying the weak axiom for a fixed distribution of wealth is a less demanding property than the invariance to redistribution property considered in Section 4.B. In particular, if the second property holds, then the first also holds, but aggregate demand (for a fixed distribution of wealth) may satisfy the weak axiom even though aggregate demand may not be invariant to redistribution of wealth (e.g., individual preferences may be homothetic but not identical).

Having spent all this time investigating the weak axiom (WA), you might ask: "What about the strong axiom (SA)?" We have not focused on the Strong Axiom for three reasons.

First, the WA is a robust property, whereas the SA (which, remember, yields the symmetry of the Slutsky matrix) is not; a priori, the chances of it being satisfied by a real economy are essentially zero. For example, if we start with a group of consumers with identical preferences and wealth, then aggregate demand obviously satisfies the SA. However, if we now perturb every preference slightly and independently across consumers, the negative semidefiniteness of the Slutsky matrices (and therefore the WA) may well be preserved but the symmetry (and therefore the SA) will almost certainly not be.

12. You may want to verify that the wealth expansion paths of Example 4.C.1 must indeed look like Figure 4.C.2(b).

13. A priori, we cannot say which form is more likely. Because the demand at zero wealth is zero, it is true that for a consumer, *some* dollar must be spent among the two goods according to shares similar to the shares of the average dollar. But if the levels of wealth are not close to zero, it does not follow that this is the case for the *marginal* dollar. It may even happen that because of incipient satiation, the shares of the marginal dollar display consumption propensities that are the reverse of the ones exhibited by the average dollar. See Hildenbrand (1994) for an account of empirical research on this matter.

Second, many of the strong positive results of general equilibrium (to be reviewed in Part IV, especially Chapters 15 and 17) to which one wishes to apply the aggregation theory discussed in this chapter depend on the weak axiom, not on the strong axiom, holding in the aggregate.

Third, while one might initially think that the existence of a preference relation explaining aggregate behavior (which is what we get from the SA) would be the condition required to use aggregate demand measures (such as aggregate consumer surplus) as welfare indicators, we will see in Section 4.D that, in fact, more than this condition is required anyway.

4.D Aggregate Demand and the Existence of a Representative Consumer

The aggregation question we pose in this section is: When can we compute meaningful measures of aggregate welfare using the aggregate demand function and the welfare measurement techniques discussed in Section 3.I for individual consumers? More specifically, when can we treat the aggregate demand function as if it were generated by a fictional *representative consumer* whose preferences can be used as a measure of aggregate societal (or *social*) welfare?

We take as our starting point a distribution rule $(w_1(p, w), \ldots, w_I(p, w))$ that to every level of aggregate wealth $w \in \mathbb{R}$ assigns individual wealths. We assume that $\sum_i w_i(p, w) = w$ for all (p, w) and that every $w_i(\cdot, \cdot)$ is continuous and homogeneous of degree one. As discussed in Sections 4.B and 4.C, aggregate demand then takes the form of a conventional market demand function $x(p, w) = \sum_i x_i(p, w_i(p, w))$. In particular, $x(p, w)$ is continuous, is homogeneous of degree zero, and satisfies Walras' law. It is important to keep in mind that the aggregate demand function $x(p, w)$ depends on the wealth distribution rule (except under the special conditions identified in Section 4.B).

It is useful to begin by distinguishing two senses in which we could say that there is a representative consumer. The first is a positive, or behavioral, sense.

Definition 4.D.1: A *positive representative consumer* exists if there is a rational preference relation $\succsim$ on $\mathbb{R}^L_+$ such that the aggregate demand function $x(p, w)$ is precisely the Walrasian demand function generated by this preference relation. That is, $x(p, w) \succ x$ whenever $x \neq x(p, w)$ and $p \cdot x \leq w$.

A positive representative consumer can thus be thought of as a fictional individual whose utility maximization problem when facing society's budget set $\{x \in \mathbb{R}^L_+: p \cdot x \leq w\}$ would generate the economy's aggregate demand function.

For it to be correct to treat aggregate demand as we did individual demand functions in Section 3.I, there must be a positive representative consumer.[14] However, although this is a necessary condition for the property of aggregate demand that we seek, it is not sufficient. We also need to be able to assign welfare significance to this

14. Note that if there is a positive representative consumer, then aggregate demand satisfies the positive properties sought in Section 4.C. Indeed, not only will aggregate demand satisfy the weak axiom, but it will also satisfy the strong axiom. Thus, the aggregation property we are after in this section is stronger than the one discussed in Section 4.C.

fictional individual's demand function. This will lead to the definition of a *normative* representative consumer. To do so, however, we first have to be more specific about what we mean by the term *social welfare*. We accomplish this by introducing the concept of a *social welfare function*, a function that provides a summary (social) utility index for any collection of individual utilities.

Definition 4.D.2: A (*Bergson-Samuelson*) *social welfare function* is a function $W: \mathbb{R}^I \to \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, \ldots, u_I) \in \mathbb{R}^I$ of utility levels for the I consumers in the economy.

The idea behind a social welfare function $W(u_1, \ldots, u_I)$ is that it accurately expresses society's judgments on how individual utilities have to be compared to produce an ordering of possible social outcomes. (We do not discuss in this section the issue of where this social preference ranking comes from. Chapters 21 and 22 cover this point in much more detail.) We also assume that social welfare functions are increasing, concave, and whenever convenient, differentiable.

Let us now hypothesize that there is a process, a benevolent central authority perhaps, that, for any given prices p and aggregate wealth level w, redistributes wealth in order to maximize social welfare. That is, for any (p, w), the wealth distribution $(w_1(p, w), \ldots, w_I(p, w))$ solves

$$\begin{aligned} \underset{w_1, \ldots, w_I}{\text{Max}} \quad & W(v_1(p, w_1), \ldots, v_I(p, w_I)) \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{I} w_i \le w, \end{aligned} \tag{4.D.1}$$

where $v_i(p, w)$ is consumer i's indirect utility function.[15,16] The optimum value of problem (4.D.1) defines a social indirect utility function $v(p, w)$. Proposition 4.D.1 shows that this indirect utility function provides a positive representative consumer for the aggregate demand function $x(p, w) = \sum_i x_i(p, w_i(p, w))$.

Proposition 4.D.1: Suppose that for each level of prices p and aggregate wealth w, the wealth distribution $(w_1(p, w), \ldots, w_I(p, w))$ solves problem (4.D.1). Then the value function $v(p, w)$ of problem (4.D.1) is an indirect utility function of a positive representative consumer for the aggregate demand function $x(p, w) = \sum_i x_i(p, w_i(p, w))$.

Proof: In Exercise 4.D.2, you are asked to establish that $v(p, w)$ does indeed have the properties of an indirect utility function. The argument for the proof then consists of using Roy's identity to derive a Walrasian demand function from $v(p, w)$, which we denote by $x_R(p, w)$, and then establishing that it actually equals $x(p, w)$.

We begin by recording the first-order conditions of problem (4.D.1) for a

15. We assume in this section that our direct utility functions $u_i(\cdot)$ are concave. This is a weak hypothesis (once quasiconcavity has been assumed) which makes sure that in all the optimization problems to be considered, the first-order conditions are sufficient for the determination of global optima. In particular, $v_i(p, \cdot)$ is then a concave function of w_i.

16. In Exercise 4.D.1, you are asked to show that if so desired, problem (4.D.1) can be equivalently formulated as one where social utility is maximized, not by distributing wealth, but by distributing bundles of goods with aggregate value at prices p not larger than w. The fact that in optimally redistributing goods, we can also restrict ourselves to redistributing wealth is, in essence, a version of the second fundamental theorem of welfare economics, which will be covered extensively in Chapter 16.

given value of (p, w). Neglecting boundary solutions, these require that for some $\lambda \geq 0$, we have

$$\lambda = \frac{\partial W}{\partial v_1}\frac{\partial v_1}{\partial w_1} = \cdots = \frac{\partial W}{\partial v_I}\frac{\partial v_I}{\partial w_I} \tag{4.D.2}$$

(For notational convenience, we have omitted the points at which the derivatives are evaluated.) Condition (4.D.2) simply says that at a socially optimal wealth distribution, the social utility of an extra unit of wealth is the same irrespective of who gets it.

By Roy's identity, we have $x_R(p, w) = -[1/(\partial v(p, w)/\partial w)]\, \nabla_p v(p, w)$. Since $v(p, w)$ is the value function of problem (4.D.1), we know that $\partial v/\partial w = \lambda$. (See Section M.K of the Mathematical Appendix) In addition, for any commodity ℓ, the chain rule and (4.D.2)—or, equivalently, the envelope theorem—give us

$$\frac{\partial v}{\partial p_\ell} = \sum_i \frac{\partial W}{\partial v_i}\frac{\partial v_i}{\partial p_\ell} + \lambda \sum_i \frac{\partial w_i}{\partial p_\ell} = \sum_i \frac{\partial W}{\partial v_i}\frac{\partial v_i}{\partial p_\ell},$$

where the second equality follows because $\sum_i w_i(p, w) = w$ for all (p, w) implies that $\sum_i (\partial w_i/\partial p_\ell) = 0$. Hence, in matrix notation, we have

$$\nabla_p v(p, w) = \textstyle\sum_i (\partial W/\partial v_i)\, \nabla_p v_i(p, w_i(p, w)).$$

Finally, using Roy's identity and the first-order condition (4.D.2), we get

$$\begin{aligned} x_R(p, w) &= -\frac{1}{\lambda}\sum_i \left[\frac{\lambda}{\partial v_i/\partial w_i}\right] \nabla_p v_i(p, w_i(p, w)) \\ &= -\sum_i \left[\frac{1}{\partial v_i/\partial w_i}\right] \nabla_p v_i(p, w_i(p, w)) \\ &= \textstyle\sum_i x_i(p, w_i(p, w)) = x(p, w), \end{aligned}$$

as we wanted to show. ■

Equipped with Proposition 4.D.1, we can now define a *normative representative consumer*.

Definition 4.D.3: The positive representative consumer $\succsim$ for the aggregate demand $x(p, w) = \sum_i x_i(p, w_i(p, w))$ is a *normative representative consumer* relative to the social welfare function $W(\cdot)$ if for every (p, w), the distribution of wealth $(w_1(p, w), \ldots, w_I(p, w))$ solves problems (4.D.1) and, therefore, the value function of problem (4.D.1) is an indirect utility function for $\succsim$.

If there is a normative representative consumer, the preferences of this consumer have welfare significance and the aggregate demand function $x(p, w)$ can be used to make welfare judgments by means of the techniques described in Section 3.I. In doing so, however, it should never be forgotten that a given wealth distribution rule [the one that solves (4.D.1) for the given social welfare function] is being adhered to and that the "level of wealth" should always be understood as the "optimally distributed level of wealth." For further discussion, see Samuelson (1956) and Chipman and Moore (1979).

Example 4.D.1: Suppose that consumers all have homothetic preferences represented by utility functions homogeneous of degree one. Consider now the social welfare function $W(u_1, \ldots, u_I) = \sum_i \alpha_i \ln u_i$ with $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. Then the optimal wealth distribution function [for problem (4.D.1)] is the price-independent rule that we adopted in Section 4.C: $w_i(p, w) = \alpha_i w$. (You are asked to demonstrate this fact in Exercise 4.D.6.) Therefore, in the homothetic case, the aggregate demand $x(p, w) = \sum_i x_i(p, \alpha_i w)$ can be viewed as originating from the normative representative consumer generated by this social welfare function. ■

Example 4.D.2: Suppose that all consumers' preferences have indirect utilities of the Gorman form $v_i(p, w_i) = a_i(p) + b(p)w_i$. Note that $b(p)$ does not depend on i, and recall that this includes as a particular case the situation in which preferences are quasilinear with respect to a common numeraire. From Section 4.B, we also know that aggregate demand $x(p, w)$ is independent of the distribution of wealth.[17]

Consider now the *utilitarian* social welfare function $\sum_i u_i$. Then *any* wealth distribution rule $(w_1(p, w), \ldots, w_I(p, w))$ solves the optimization problem (4.D.1), and the indirect utility function that this problem generates is simply $v(p, w) = \sum_i a_i(p) + b(p)w$. (You are asked to show these facts in Exercise 4.D.7.) One conclusion is, therefore, that when indirect utility functions have the Gorman form [with common $b(p)$] and the social welfare function is utilitarian, then aggregate demand can *always* by viewed as being generated by a normative representative consumer.

When consumers have Gorman-form indirect utility functions [with common $b(p)$], the theory of the normative representative consumer admits an important strengthening. In general, the preferences of the representative consumer depend on the form of the social welfare function. *But not in this case.* We now verify that if the indirect utility functions of the consumers have the Gorman form [with common $b(p)$], then the preferences of the representative consumer are independent of the particular social welfare function used.[18] In fact, we show that $v(p, w) = \sum_i a_i(p) + b(p)w$ is an admissible indirect utility function for the normative representative consumer relative to *any* social welfare function $W(u_1, \ldots, u_I)$.

To verify this claim, consider a particular social welfare function $W(\cdot)$, and denote the value function of problem (4.D.1), relative to $W(\cdot)$, by $v^*(p, w)$. We must show that the ordering induced by $v(\cdot)$ and $v^*(\cdot)$ is the same, that is, that for any pair (p, w) and (p', w') with $v(p, w) < v(p', w')$, we have $v^*(p, w) < v^*(p', w')$. Take the vectors of individual wealths $(w_1, \ldots, w_I)$ and $(w'_1, \ldots, w'_I)$ reached as optima of (4.D.1), relative to $W(\cdot)$, for (p, w) and (p', w'), respectively. Denote $u_i = a_i(p) + b(p)w_i$, $u'_i = a_i(p) + b(p)w'_i$, $u = (u_1, \ldots, u_I)$, and $u' = (u'_1, \ldots, u'_I)$. Then $v^*(p, w) = W(u)$ and $v^*(p', w') = W(u')$. Also $v(p, w) = \sum_i a_i(p) + b(p)w = \sum_i u_i$, and similarly, $v(p', w') = \sum_i u'_i$. Therefore, $v(p, w) < v(p', w')$ implies $\sum_i u_i < \sum_i u'_i$. We argue that $\nabla W(u') \cdot (u - u') < 0$, which, $W(\cdot)$ being concave, implies the desired result, namely $W(u) < W(u')$.[19] By expression (4.D.2), at an optimum we have $(\partial W/\partial v_i)(\partial v_i/\partial w_i) = \lambda$ for all i. But in our case, $\partial v_i/\partial w_i = b(p)$ for all i. Therefore, $\partial W/\partial v_i = \partial W/\partial v_j > 0$ for any i, j. Hence, $\sum_i u_i < \sum_i u'_i$ implies $\nabla W(u') \cdot (u - u') < 0$.

The previous point can perhaps be better understood if we observe that when

17. As usual, we neglect the nonnegativity constraints on consumption.

18. But, of course, the optimal distribution rules will typically depend on the social welfare function. Only for the utilitarian social welfare function will it not matter how wealth is distributed.

19. Indeed, concavity of $W(\cdot)$ implies $W(u') + \nabla W(u') \cdot (u - u') \geq W(u)$; see Section M.C of the Mathematical Appendix.

preferences have the Gorman form [with common $b(p)$], then (p', w') is socially better than (p, w) for the utilitarian social welfare function $\sum_i u_i$ *if and only if* when compared with (p, w), (p', w') passes the following *potential compensation test*: For any distribution $(w_1, \ldots, w_I)$ of w, there is a distribution $(w'_1, \ldots, w'_I)$ of w' such that $v_i(p', w'_i) > v_i(p, w_i)$ for all i. To verify this is straightforward. Suppose that

$$(\textstyle\sum_i a_i(p') + b(p')w') - (\textstyle\sum_i a_i(p) + b(p)w) = c > 0.$$

Then the wealth levels w'_i implicitly defined by $a_i(p') + b(p')w'_i = a_i(p) + b(p)w_i + c/I$ will be as desired.[20] Once we know that (p', w') when compared with (p, w) passes the potential compensation test, it follows merely from the definition of the optimization problem (4.D.1) that (p', w') is better than (p, w) for any normative consumer, that is, for any social welfare function that we may wish to employ (see Exercise 4.D.8).

The two properties just presented—independence of the representative consumer's preferences from the social welfare function and the potential compensation criterion will be discussed further in Sections 10.F and 22.C. For the moment, we simply emphasize that they are not general properties of normative representative consumers. By choosing the distribution rules that solve (4.D.1), we can generate a normative representative consumer for any set of individual utilities and any social welfare function. For the properties just reviewed to hold, the individual preferences have been required to have the Gorman form [with common $b(p)$]. ■

It is important to stress the distinction between the concepts of a positive and a normative representative consumer. It is *not* true that whenever aggregate demand can be generated by a positive representative consumer, this representative consumer's preferences have normative content. It may even be the case that a positive representative consumer exists but that there is *no* social welfare function that leads to a normative representative consumer. We expand on this point in the next few paragraphs [see also Dow and Werlang (1988) and Jerison (1994)].

We are given a distribution rule $(w_1(p, w), \ldots, w_I(p, w))$ and assume that a positive representative consumer with utility function $u(x)$ exists for the aggregate demand $x(p, w) = \sum_i x_i(p, w_i(p, w))$. In principle, using the integrability techniques presented in Section 3.H, it should be possible to determine the preferences of the representative consumer from the knowledge of $x(p, w)$. Now fix any $(\bar{p}, \bar{w})$, and let $\bar{x} = x(\bar{p}, \bar{w})$. Relative to the aggregate consumption vector $\bar{x}$, we can define an at-least-as-good-as set for the representative consumer:

$$B = \{x \in \mathbb{R}^L_+ : u(x) \geq (\bar{x})\} \subset \mathbb{R}^L_+.$$

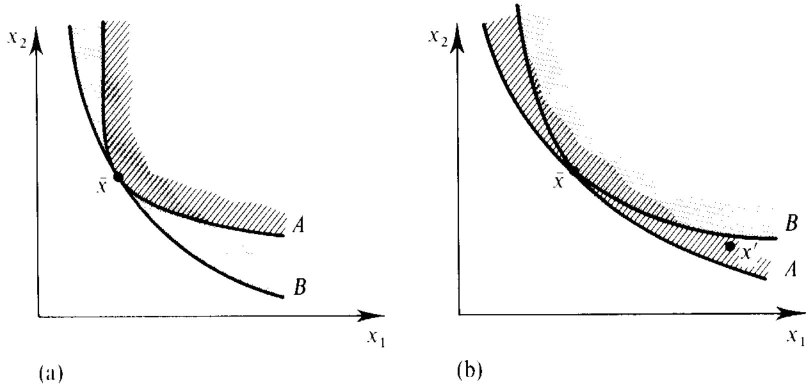
Next, let $\bar{w}_i = w_i(\bar{p}, \bar{w})$ and $\bar{x}_i = x_i(\bar{p}, \bar{w}_i)$, and consider the set

$$A = \{x = \textstyle\sum_i x_i : x_i \succsim_i \bar{x}_i \text{ for all } i\} \subset \mathbb{R}^L_+.$$

In words, A is the set of aggregate consumption vectors for which there is a distribution of commodities among consumers that makes every consumer as well off as under $(\bar{x}_1, \ldots, \bar{x}_I)$. The boundary of this set is sometimes called a *Scitovsky contour*. Note that both set A and set B are supported by the price vector $\bar{p}$ at $\bar{x}$ (see Figure 4.D.1).

If the given wealth distribution comes from the solution to a social welfare optimization problem of the type (4.D.1) (i.e., if the positive representative consumer is in fact a normative

20. We continue to neglect nonnegativity constraints on wealth.



$A = \{x \in \sum_i x_i : u_i(x_i) \geq u_i(\bar{x}_i) \text{ for all } i\}$
$B = \{x \in \mathbb{R}^2_+ : u(x) \geq u(\bar{x})\}$

Figure 4.D.1
Comparing the at-least-as-good-as set of the positive representative consumer with the sum of the at-least-as-good-as sets of the individual consumers.
(a) The positive representative consumer could be a normative representative consumer.
(b) The positive representative consumer cannot be a normative representative consumer.

representative consumer), then this places an important restriction on how sets A and B relate to each other: Every element of set A must be an element of set B. This is so because the social welfare function underlying the normative representative consumer is increasing in the utility level of every consumer (and thus any aggregate consumption bundle that could be distributed in a manner that guarantees to every consumer a level of utility as high as the levels corresponding to the optimal distribution of $\bar{x}$ must receive a social utility higher than the latter; see Exercise 4.D.4). That is, a *necessary* condition for the existence of a normative representative consumer is that $A \subset B$. A case that satisfies this necessary condition is depicted in Figure 4.D.1(a).

However, there is nothing to prevent the existence, in a particular setting, of a positive representative consumer with a utility function $u(x)$ that fails to satisfy this condition, as in Figure 4.D.1(b). To provide some further understanding of this point, Exercise 4.D.9 asks you to show that $A \subset B$ implies that $\sum_i S_i(\bar{p}, \bar{w}_i) - S(\bar{p}, \bar{w})$ is positive semidefinite, where $S(p, w)$ and $S_i(p, w_i)$ are the Slutsky matrices of aggregate and individual demand, respectively. Informally, we could say that the substitution effects of aggregate demand must be larger in absolute value than the sum of individual substitution effects (geometrically, this corresponds to the boundary of B being flatter at $\bar{x}$ than the boundary of A). This observation allows us to generate in a simple manner examples in which aggregate demand can be rationalized by preferences but, nonetheless, there is no normative representative consumer.

Suppose, for example, that the wealth distribution rule is of the form $w_i(p, w) = \alpha_i w$. Suppose also that $S(p, w)$ happens to be symmetric for all (p, w); if $L = 2$, this is automatically satisfied. Then, from integrability theory (see Section 3.H), we know that a sufficient condition for the existence of underlying preferences is that, for all (p, w), we have $dp \cdot S(p, w)\, dp < 0$ for all $dp \neq 0$ not proportional to p (we abbreviate this as the *n.d. property*). On the other hand, as we have just seen, a necessary condition for the existence of a normative representative consumer is that $C(\bar{p}, \bar{w}) = \sum_i S_i(\bar{p}, \bar{w}_i) - S(\bar{p}, \bar{w})$ be positive semidefinite [this is the same matrix discussed in Section 4.C; see expression (4.C.8)]. Thus, if $S(p, w)$ has the n.d. property for all (p, w) but $C(\bar{p}, \bar{w})$ is not positive semidefinite [i.e., wealth effects are such that $S(\bar{p}, \bar{w})$ is "less negative" than $\sum_i S_i(\bar{p}, \bar{w})$], then a positive representative consumer exists that, nonetheless, cannot be made normative for any social welfare function. (Exercise 4.D.10 provides an instance where this is indeed the case.) In any example of this nature we have moves in aggregate consumption that would pass a potential compensation test (each consumer's welfare could be made better off by an appropriate distribution of the move) but are regarded as socially inferior under the utility function that rationalizes aggregate demand. [In Figure 4.D.1(b), this could be the move from $\bar{x}$ to x'.]

The moral of all this is clear: The existence of preferences that explain behavior is not enough to attach to them any welfare significance. For the latter, it is also necessary that these preferences exist for the right reasons. ■

APPENDIX A: REGULARIZING EFFECTS OF AGGREGATION

This appendix is devoted to making the point that although aggregation can be deleterious to the preservation of the good properties of individual demand, it can also have helpful *regularizing* effects. By regularizing, we mean that the average (per-consumer) demand will tend to be more continuous or smooth, as a function of prices, than the individual components of the sum.

Recall that if preferences are strictly convex, individual demand functions are continuous. As we noted, aggregate demand will then be continuous as well. But average demand can be (nearly) continuous even when individual demands are not. The key requirement is one of *dispersion* of individual preferences.

Example 4.AA.1: Suppose that there are two commodities. Consumers have quasilinear preferences with the second good as numeraire. The first good, on the other hand, is available only in integer amounts, and consumers have no wish for more than one unit of it. Thus, normalizing the utility of zero units of the first good to be zero, the preferences of consumer i are completely described by a number v_{1i}, the utility in terms of numeraire of holding one unit of the first good. It is then clear that the demand for the first good by consumer i is given by the correspondence

$$\begin{aligned} x_{1i}(p_1) &= 1 \quad &\text{if } p_1 < v_{1i}, \\ &= \{0, 1\} \quad &\text{if } p_1 = v_{1i}, \\ &= 0 \quad &\text{if } p_1 > v_{1i}, \end{aligned}$$

which is depicted in Figure 4.AA.1(a). Thus, individual demand exhibits a sudden, discontinuous jump in demand from 0 to 1 as the price crosses the value $p_1 = v_{1i}$.

Suppose now that there are many consumers. In fact, consider the limit situation where there is an actual continuum of consumers. We could then say that individual preferences are *dispersed* if there is no concentrated group of consumers having any particular value of v_1 or, more precisely, if the statistical distribution function of the v_1's, $G(v_1)$, is continuous. Then, denoting by $x_1(p_1)$ the average demand for the first good, we have $x_1(p_1) =$ "mass of consumers with $v_1 > p_1$" $= 1 - G(p_1)$.

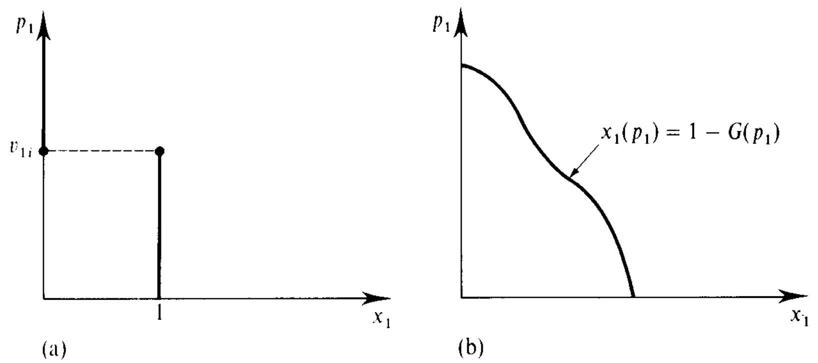


Figure 4.AA.1
The regularizing effect of aggregation.
(a) Individual demand.
(b) Aggregate demand when the distribution of the v_1's is $G(\cdot)$.

Hence, the aggregate demand $x_1(\cdot)$, shown in Figure 4.AA.1(b), is a nice continuous function even though none of the individual demand correspondences are so. Note that with only a finite number of consumers, the distribution function $G(\cdot)$ cannot quite be a continuous function; but if the consumers are many, then it can be nearly continuous. ■

The regularizing effects of aggregation are studied again in Section 17.I. We show there that in general (i.e., without dispersedness requirements), the aggregation of numerous individual demand correspondences will generate a (nearly) *convex-valued* average demand correspondence.

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EXERCISES

4.B.1[B] Prove the sufficiency part of Proposition 4.B.1. Show also that if preferences admit Gorman-form indirect utility functions with the same $b(p)$, then preferences admit expenditure functions of the form $e_i(p, u_i) = c(p)u_i + d_i(p)$.

4.B.2[B] Suppose that there are I consumers and L commodities. Consumers differ only by their wealth levels w_i and by a taste parameter s_i, which we could call *family size*. Thus, denote the indirect utility function of consumer i by $v(p, w_i, s_i)$. The corresponding Walrasian demand function for consumer i is $x(p, w_i, s_i)$.

(a) Fix $(s_1, \ldots, s_I)$. Show that if for any $(w_1, \ldots, w_I)$ aggregate demand can be written as a function of only p and aggregate wealth $w = \sum_i w_i$ (or, equivalently, average wealth), and if every consumer's preference relationship $\succsim_i$ is homothetic, then all these preferences must be identical [and so $x(p, w_i, s_i)$ must be independent of s_i].

(b) Give a sufficient condition for aggregate demand to depend only on aggregate wealth w and $\sum_i s_i$ (or, equivalently, average wealth and average family size).

4.C.1^C Prove that if $x_i(p, w_i)$ satisfies the ULD, then $D_p x_i(p, w_i)$ is negative semidefinite [i.e., $dp \cdot D_p x_i(p, w_i)\, dp \leq 0$ for all dp]. Also show that if $D_p x_i(p, w_i)$ is negative definite for all p, then $x_i(p, w_i)$ satisfies the ULD (this second part is harder).

4.C.2^A Prove a version of Proposition 4.C.1 by using the (sufficient) differential versions of the ULD and the WA. (Recall from the small type part of Section 2.F that a sufficient condition for the WA is that $v \cdot S(p, w) v < 0$ whenever v is not proportional to p.)

4.C.3^A Give a graphical two-commodity example of a preference relation generating a Walrasian demand that does not satisfy the ULD property. Interpret.

4.C.4^C Show that if the preference relation $\succsim_i$ on $\mathbb{R}^2_+$ has L-shaped indifference curves and the demand function $x_i(p, w_i)$ has the ULD property, then $\succsim_i$ must be homothetic. [*Hint*: The L shape of indifference curves implies $S_i(p, w_i) = 0$ for all (p, w_i); show that if $D_{w_i} x_i(\bar{p}, \bar{w}_i) \neq (1/\bar{w}_i) x_i(\bar{p}, \bar{w}_i)$, then there is $v \in \mathbb{R}^L$ such that $v \cdot D_p x_i(\bar{p}, \bar{w}_i) v > 0$.]

4.C.5^C Prove Proposition 4.C.3. To that effect, you can fix $w = 1$. The proof is best done in terms of the indirect demand function $g_i(x) = (1/x \cdot \nabla u_i(x))\, \nabla u_i(x)$ [note that $x = x_i(g_i(x), 1)$]. For an individual consumer, the ULD is self-dual; that is, it is equivalent to $(g_i(x) - g_i(y)) \cdot (x - y) < 0$ for all $x \neq y$. In turn, this property is implied by the negative definiteness of $Dg_i(x)$ for all x. Hence, concentrate on proving this last property. More specifically, let $v \neq 0$, and denote $q = \nabla u_i(x)$ and $C = D^2 u_i(x)$. You want to prove $v \cdot Dg_i(x) v < 0$. [*Hint*: You can first assume $q \cdot v = q \cdot x$; then differentiate $g_i(x)$, and make use of the equality $v \cdot Cv - x \cdot Cv = (v - \frac{1}{2}x) \cdot C(v - \frac{1}{2}x) - \frac{1}{4} x \cdot Cx$.]

4.C.6^A Show that if $u_i(x_i)$ is homogeneous of degree one, so that $\succsim_i$ is homothetic, then $\sigma_i(x_i) = 0$ for all x_i [$\sigma_i(x_i)$ is the quotient defined in Proposition 4.C.3].

4.C.7^B Show that Proposition 4.C.4 still holds if the distribution of wealth has a nonincreasing density function on $[0, \bar{w}]$. A more realistic distribution of wealth would be *unimodal* (i.e., an increasing and then decreasing density function with a single peak). Argue that there are unimodal distributions for which the conclusions of the proposition do not hold.

4.C.8^A Derive expression (4.C.7), the aggregate version of the Slutsky matrix.

4.C.9^A Verify that if individual preferences $\succsim_i$ are homothetic, then the matrix $C(p, w)$ defined in expression (4.C.8) is positive semidefinite.

4.C.10^C Argue that for the Hildenbrand example studied in Proposition 4.C.4, $C(p, w)$ is positive semidefinite. Conclude that aggregate demand satisfies the WA for that wealth distribution. [Note: You must first adapt the definition of $C(p, w)$ to the continuum-of-consumers situation of the example.]

4.C.11^B Suppose there are two consumers, 1 and 2, with utility functions over two goods, 1 and 2, of $u_1(x_{11}, x_{21}) = x_{11} + 4\sqrt{x_{21}}$ and $u_2(x_{12}, x_{22}) = 4\sqrt{x_{12}} + x_{22}$. The two consumers have identical wealth levels $w_1 = w_2 = w/2$.

(a) Calculate the individual demand functions and the aggregate demand function.

(b) Compute the individual Slutsky matrices $S_i(p, w/2)$ (for $i = 1, 2$) and the aggregate

Slutsky matrix $S(p, w)$. [*Hint*: Note that for this two-good example, only one element of each matrix must be computed to determine the entire matrix.] Show that $dp \cdot S(p, w)\, dp < 0$ for all $dp \neq 0$ not proportional to p. Conclude that aggregate demand satisfies the *WA*.

(c) Compute the matrix $C(p, w) = \sum_i S_i(p, w/2) - S(p, w)$ for prices $p_1 = p_2 = 1$. Show that it is positive semidefinite if $w > 16$ and that it is negative semidefinite if $8 < w < 16$. In fact, argue that in the latter case, $dp \cdot C(p, w)\, dp < 0$ for some dp [so that $C(p, w)$ is *not* positive semidefinite]. Conclude that $C(p, w)$ positive semidefinite is not necessary for the WA to be satisfied.

(d) For each of the two cases $w > 16$ and $8 < w < 16$, draw a picture in the (x_1, x_2) plane depicting each consumer's consumption bundle and his wealth expansion path for the prices $p_1 = p_2 = 1$. Compare your picture with Figure 4.C.2.

4.C.12^{B} The results presented in Sections 4.B and 4.C indicate that if for any $(w_1, \ldots, w_I)$ aggregate demand can be written as a function of only aggregate wealth [i.e., as $x(p, \sum_i w_i)$], then aggregate demand must satisfy the WA. The *distribution function* $F\colon [0, \infty) \to [0, 1]$ of $(w_1, \ldots, w_I)$ is defined as $F(w) = (1/I)$(*number of* i's with $w_i \leq w$) for any w. Suppose now that for any $(w_1, \ldots, w_I)$, aggregate demand can be written as a function of the corresponding aggregate *distribution* $F(\cdot)$ of wealth. Show that aggregate demand does not necessarily satisfy the WA. [*Hint*: It suffices to give a two-commodity, two-consumer example where preferences are identical, wealths are $w_1 = 1$ and $w_2 = 3$, and the WA fails. Try to construct the example graphically. It is a matter of making sure that four suitably positioned indifference curves can be fitted together without crossing.]

4.C.13^{C} Consider a two-good environment with two consumers. Let the wealth distribution rule be $w_1(p, w) = wp_1/(p_1 + p_2)$, $w_2(p, w) = wp_2/(p_1 + p_2)$. Exhibit an example in which the two consumers have homothetic preferences but, nonetheless, the aggregate demand fails to satisfy the weak axiom. A good picture will suffice. Why does not Proposition 4.C.1 apply?

4.D.1^{B} In this question we are concerned with a normative representative consumer. Denote by $v(p, w)$ the optimal value of problem (4.D.1), and by $(w_1(p, w), \ldots, w_I(p, w))$ the corresponding optimal wealth distribution rules. Verify that $v(p, w)$ is also the optimal value of

$$\underset{x_1, \ldots, x_I}{\text{Max}} \quad W(u_1(x_1), \ldots, u_I(x_I)) \qquad \text{s.t. } p \cdot (\textstyle\sum_i x_i) \leq w$$

and that $[x_1(p, w_1(p, w)), \ldots, x_I(p, w_I(p, w))]$ is a solution to this latter problem. Note the implication that to maximize social welfare given prices p and wealth w, the planner need not control consumption directly, but rather need only distribute wealth optimally and allow consumers to make consumption decisions independently given prices p.

4.D.2^{B} Verify that $v(p, w)$, defined as the optimal value of problem (4.D.1), has the properties of an indirect utility function (i.e., that it is homogeneous of degree zero, increasing in w and decreasing in p, and quasiconvex).

4.D.3^{B} It is good to train one's hand in the use of inequalities and the Kuhn–Tucker conditions. Prove Proposition 4.D.1 again, this time allowing for corner solutions.

4.D.4^{C} Suppose that there is a normative representative consumer with wealth distribution rule $(w_1(p, w), \ldots, w_I(p, w))$. For any $x \in \mathbb{R}^L_+$, define

$$u(x) = \underset{(x_1, \ldots, x_I)}{\text{Max}} \quad W(u_1(x_1), \ldots, u_I(x_I)) \qquad \text{s.t. } \textstyle\sum_i x_i \leq x.$$

(a) Give conditions implying that $u(\cdot)$ has the properties of a utility function; that is, it is monotone, continuous, and quasiconcave (and even concave).

(b) Show that for any (p, w), the Walrasian demand generated from the problem $\text{Max}_x\ u(x)$ s.t. $p \cdot x \leq w$ is equal to the aggregate demand function.

4.D.5[A] Suppose that there are I consumers and that consumer i's utility function is $u_i(x_i)$, with demand function $x_i(p, w_i)$. Consumer i's wealth w_i is generated according to a wealth distribution rule $w_i = \alpha_i w$, where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. Provide an example (i.e., a set of utility functions) in which this economy does *not* admit a positive representative consumer.

4.D.6[B] Establish the claims made in Example 4.D.1.

4.D.7[B] Establish the claims made in the second paragraph of Example 4.D.2.

4.D.8[A] Say that (p', w') passes the *potential compensation test* over (p, w) if for any distribution $(w_1, \dots, w_I)$ of w there is a distribution $(w'_1, \dots, w'_I)$ of w' such that $v_i(p', w'_i) > v_i(p, w_i)$ for all i. Show that if $(p'w')$ passes the potential compensation test over (p, w), any normative representative consumer must prefer (p', w') over (p, w).

4.D.9[B] Show that $A \subset B$ (notation as in Section 4.D) implies that $S(\bar{p}, \bar{w}) - \sum_i S_i(\bar{p}, \bar{w}_i)$ is negative semidefinite. [*Hint*: Consider $g(p) = e(p, u(\bar{x})) - \sum_i e_i(p, u_i(\bar{x}_i))$, where $e(\cdot)$ is the expenditure function for $u(\cdot)$ and $e_i(\cdot)$ is the expenditure function for $u_i(\cdot)$. Note that $A = \sum_i \{x_i : u_i(x_i) \geq u_i(\bar{x}_i)\}$ implies that $\sum_i e_i(p, u_i(\bar{x}_i))$ is the optimal value of the problem $\text{Min}_{x \in A}\ p \cdot x$. From this and $A \subset B$, you get $g(p) \leq 0$ for all p and $g(\bar{p}) = 0$. Therefore, $D^2 g(\bar{p})$ is negative semidefinite. Show then that $D^2 g(\bar{p}) = S(\bar{p}, \bar{w}) - \sum_i S_i(\bar{p}, \bar{w}_i)$.]

4.D.10[A] Argue that in the example considered in Exercise 4.C.11, there is a positive representative consumer rationalizing aggregate demand but that there cannot be a normative representative consumer.

4.D.11[C] Argue that for $L > 2$, the Hildenbrand case of Proposition 4.C.4 need not admit a positive representative consumer. [*Hint*: Argue that the Slutsky matrix may fail to be symmetric.]