Dynamic Games

9.A Introduction

In Chapter 8, we studied simultaneous-move games. Most economic situations, however, involve players choosing actions over time.[1] For example, a labor union and a firm might make repeated offers and counteroffers to each other in the course of negotiations over a new contract. Likewise, firms in a market may invest today in anticipation of the effects of these investments on their competitive interactions in the future. In this chapter, we therefore shift our focus to the study of *dynamic games*.

One way to approach the problem of prediction in dynamic games is to simply derive their normal form representations and then apply the solution concepts studied in Chapter 8. However, an important new issue arises in dynamic games: the *credibility* of a player's strategy. This issue is the central concern of this chapter.

Consider a vivid (although far-fetched) example: You walk into class tomorrow and your instructor, a sane but very enthusiastic game theorist, announces, "This is an important course, and I want exclusive dedication. Anyone who does not drop every other course will be barred from the final exam and will therefore flunk." After a moment of bewilderment and some mental computation, your first thought is, "Given that I indeed prefer this course to all others, I had better follow her instructions" (after all, you have studied Chapter 8 carefully and know what a best response is). But after some further reflection, you ask yourself, "Will she really bar me from the final exam if I do not obey? This is a serious institution, and she will surely lose her job if she carries out the threat." You conclude that the answer is "no" and refuse to drop the other courses, and indeed, she ultimately does not bar you from the exam. In this example, we would say that your instructor's announced strategy, "I will bar you from the exam if you do not drop every other course," is not credible. Such empty threats are what we want to rule out as equilibrium strategies in dynamic games.

In Section 9.B, we demonstrate that the Nash equilibrium concept studied in Chapter 8 does not suffice to rule out noncredible strategies. We then introduce a stronger solution concept, known as *subgame perfect Nash equilibrium*, that helps

1. As do most parlor games.

to do so. The central idea underlying this concept is the *principle of sequential rationality*: equilibrium strategies should specify optimal behavior from any point in the game onward, a principle that is intimately related to the procedure of *backward induction.*

In Section 9.C, we show that the concept of subgame perfection is not strong enough to fully capture the idea of sequential rationality in games of imperfect information. We then introduce the notion of a *weak perfect Bayesian equilibrium* (also known as a *weak sequential equilibrium*) to push the analysis further. The central feature of a weak perfect Bayesian equilibrium is its explicit introduction of a player's *beliefs* about what may have transpired prior to her move as a means of testing the sequential rationality of the player's strategy. The modifier *weak* refers to the fact that the weak perfect Bayesian equilibrium concept imposes a *minimal* set of consistency restrictions on players' beliefs. Because the weak perfect Bayesian equilibrium concept can be too weak, we also examine some related equilibrium notions that impose stronger consistency restrictions on beliefs, discussing briefly stronger notions of *perfect Bayesian equilibrium* and, in somewhat greater detail, the concept of *sequential equilibrium.*

In Section 9.D, we go yet further by asking whether certain beliefs can be regarded as "unreasonable" in some situations, thereby allowing us to further refine our predictions. This leads us to consider the notion of *forward induction.*

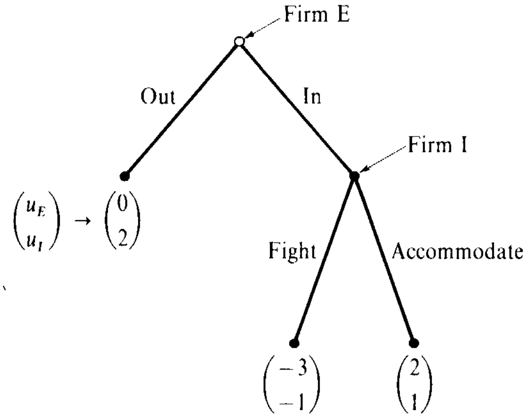
Appendix A studies finite and infinite horizon models of bilateral bargaining as an illustration of the use of subgame perfect Nash equilibrium in an important economic application. Appendix B extends the discussion in Section 9.C by examining the notion of an *extensive form trembling-hand perfect Nash equilibrium.*

We should note that—following most of the literature on this subject—all the analysis in this chapter consists of attempts to "refine" the concept of Nash equilibrium; that is, we take the position that we want our prediction to be a Nash equilibrium, and we then propose additional conditions for such an equilibrium to be a "satisfactory" prediction. However, the issues that we discuss here are not confined to this approach. We might, for example, be concerned about noncredible strategies even if we were unwilling to impose the mutually correct expectations condition of Nash equilibrium and wanted to focus instead only on rationalizable outcomes. See Bernheim (1984) and, especially, Pearce (1984) for a discussion of nonequilibrium approaches to these issues.

9.B Sequential Rationality, Backward Induction, and Subgame Perfection

We begin with an example to illustrate that in dynamic games the Nash equilibrium concept may not give sensible predictions. This observation leads us to develop a strengthening of the Nash equilibrium concept known as *subgame perfect Nash equilibrium.*

Example 9.B.1: Consider the following *predation* game. Firm E (for entrant) is considering entering a market that currently has a single incumbent (firm I). If it does so (playing "in"), the incumbent can respond in one of two ways: It can either accommodate the entrant, giving up some of its sales but causing no change in



		Firm I	
		Fight if Firm E Plays "In"	Accommodate if Firm E Plays "In"
Firm E	Out	0, 2	0, 2
	In	−3, −1	2, 1

Figure 9.B.1 Extensive and normal forms for Example 9.B.1. The Nash equilibrium (σ_E, σ_I) = (out, fight if firm E plays "in") involves a noncredible threat.

the market price, or it can fight the entrant, engaging in a costly war of predation that dramatically lowers the market price. The extensive and normal form representations of this game are depicted in Figure 9.B.1.

Examining the normal form, we see that this game has two pure strategy Nash equilibria: (σ_E, σ_I) = (out, fight if firm E plays "in") and (σ_E, σ_I) = (in, accommodate if firm E plays "in"). Consider the first of these strategy profiles. Firm E prefers to stay out of the market if firm I will fight after it enters. On the other hand, "fight if firm E plays 'in'" is an optimal choice for the incumbent if firm E is playing "out." Similar arguments show that the second pair of strategies is also a Nash equilibrium.

Yet, we claim that (out, fight if firm E plays "in") is not a sensible prediction for this game. As in the example of your instructor that we posed in Section 9.A, firm E can foresee that if it does enter, the incumbent will, in fact, find it optimal to accommodate (by doing so, firm I earns 1 rather than -1). Hence, the incumbent's strategy "fight if firm E plays 'in'" is not credible. ∎

Example 9.B.1 illustrates a problem with the Nash equilibrium concept in dynamic games. In this example, the concept is, in effect, permitting the incumbent to make an empty threat that the entrant nevertheless takes seriously when choosing its strategy. The problem with the Nash equilibrium concept here arises from the fact that when the entrant plays "out," actions at decision nodes that are unreached by play of the equilibrium strategies (here, firm I's action at the decision node following firm E's unchosen move "in") do not affect firm I's payoff. As a result, firm I can plan to do *absolutely anything* at this decision node: Given firm E's strategy of choosing "out," firm I's payoff is still maximized. *But*—and here is the crux of the matter—what firm I's strategy says it will do at the unreached node can actually *insure* that firm E, taking firm I's strategy as given, wants to play "out."

To rule out predictions such as (out, fight if firm E plays "in"), we want to insist that players' equilibrium strategies satisfy what might be called the *principle of sequential rationality*: A player's strategy should specify optimal actions *at every point in the game tree*. That is, given that a player finds herself at some point in the tree, her strategy should prescribe play that is optimal from that point on given her opponents' strategies. Clearly, firm I's strategy "fight if firm E plays 'in'" does not: after entry, the only optimal strategy for firm I is "accommodate."

In Example 9.B.1, there is a simple procedure that can be used to identify the

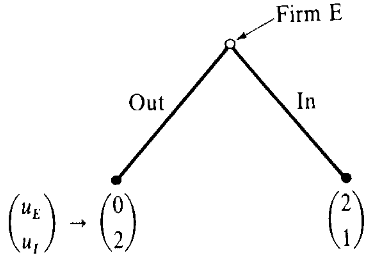


Figure 9.B.2
Reduced game after solving for post-entry behavior in Example 9.B.1.

desirable (i.e., sequentially rational) Nash equilibrium (σ_E, σ_I) = (in, accommodate if firm E plays "in"). We first determine optimal behavior for firm I in the post-entry stage of the game; this is "accommodate." Once we have done this, we then determine firm E's optimal behavior earlier in the game given the anticipation of what will happen after entry. Note that this second step can be accomplished by considering a *reduced* extensive form game in which firm I's post-entry decision is replaced by the payoffs that will result from firm I's optimal post-entry behavior. See Figure 9.B.2. This reduced game is a very simple single-player decision problem in which firm E's optimal decision is to play "in." In this manner, we identify the sequentially rational Nash equilibrium strategy profile (σ_E, σ_I) = (in, accommodate if firm E plays "in").

This type of procedure, which involves solving first for optimal behavior at the "end" of the game (here, at the post-entry decision node) and then determining what optimal behavior is earlier in the game given the anticipation of this later behavior, is known as *backward induction* (or *backward programming*). It is a procedure that is intimately linked to the idea of sequential rationality because it insures that players' strategies specify optimal behavior at every decision node of the game.

The game in Example 9.B.1 is a member of a general class of games in which the backward induction procedure can be applied to capture the idea of sequential rationality with great generality and power: *finite games of perfect information.* These are games in which every information set contains a single decision node and there is a finite number of such nodes (see Chapter 7).[2] Before introducing a formal equilibrium concept, we first discuss the general application of the backward induction procedure to this class of games.

Backward Induction in Finite Games of Perfect Information

To apply the idea of backward induction in finite games of perfect information, we start by determining the optimal actions for moves at the final decision nodes in the tree (those for which the only successor nodes are terminal nodes). Just as in firm I's post-entry decision in Example 9.B.1, play at these nodes involves no further strategic interactions among the players, and so the determination of optimal behavior at these decision nodes involves a simple single-person decision problem. Then, given that these will be the actions taken at the final decision nodes, we can proceed to the next-to-last decision nodes and determine the optimal actions to be

2. The assumption of finiteness is important for some aspects of this analysis. We discuss this point further toward the end of the section.

taken there by players who correctly anticipate the actions that will follow at the final decision nodes, and so on backward through the game tree.

This procedure is readily implemented using reduced games. At each stage, after solving for the optimal actions at the current final decision nodes, we can derive a new reduced game by deleting the part of the game following these nodes and assigning to these nodes the payoffs that result from the already determined continuation play.

Example 9.B.2: Consider the three-player finite game of perfect information depicted in Figure 9.B.3(a). The arrows in Figure 9.B.3(a) indicate the optimal play at the final decision nodes of the game. Figure 9.B.3(b) is the reduced game formed by replacing these final decision nodes by the payoffs that result from optimal play once these nodes have been reached. Figure 9.B.3(c) represents the reduced game derived

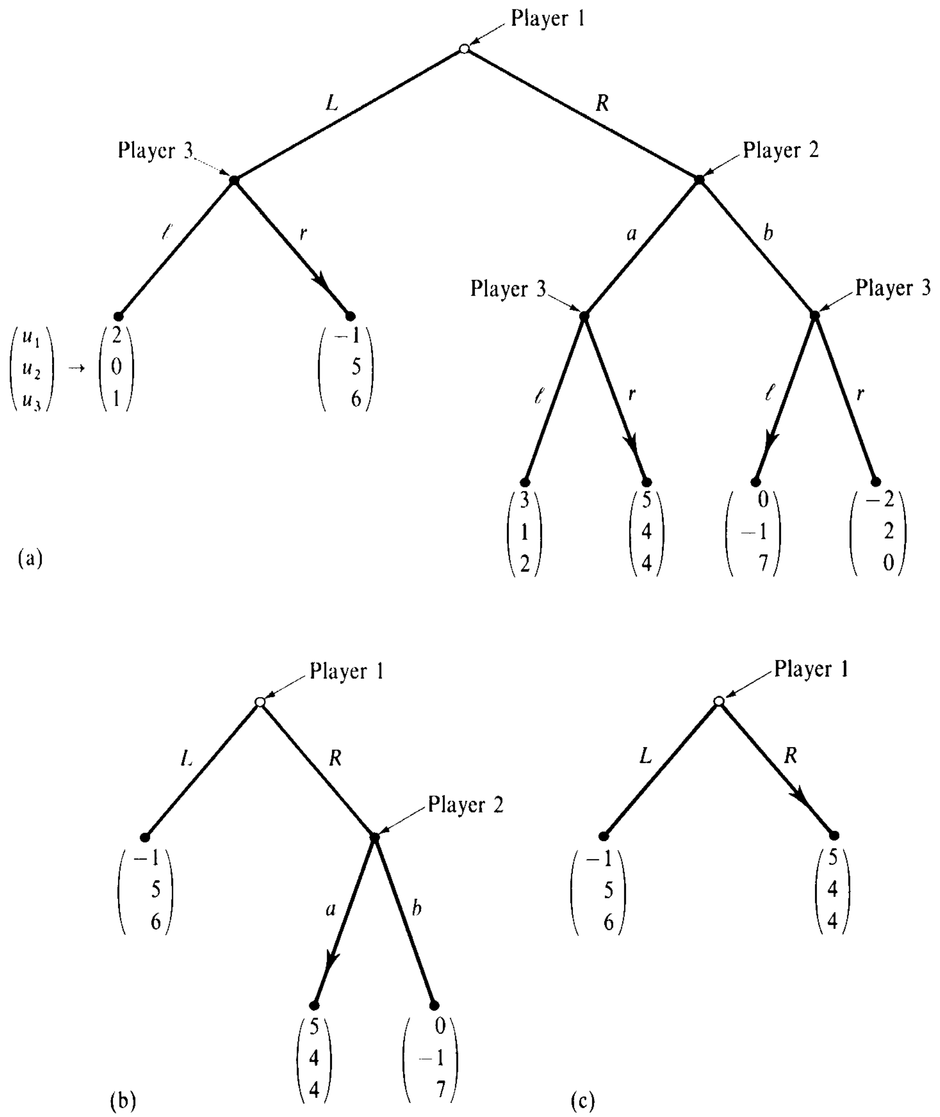


Figure 9.B.3
Reduced games in a backward induction procedure for a finite game of perfect information.
(a) Original game.
(b) First reduced game. (c) Second reduced game.

in the next stage of the backward induction procedure, when the final decision nodes of the reduced game in Figure 9.B.3(b) are replaced by the payoffs arising from optimal play at these nodes (again indicated by arrows). The backward induction procedure therefore identifies the strategy profile $(\sigma_1, \sigma_2, \sigma_3)$ in which $\sigma_1 = R, \sigma_2 =$ "a if player 1 plays R," and

$$\sigma_3 = \begin{cases} r \text{ if player 1 plays } L \\ r \text{ if player 1 plays } R \text{ and player 2 plays } a \\ \ell \text{ if player 1 plays } R \text{ and player 2 plays } b. \end{cases}$$

Note that this strategy profile is a Nash equilibrium of this three-player game but that the game also has other pure strategy Nash equilibria. (Exercise 9.B.3 asks you to verify these two points and to argue that these other Nash equilibria do not satisfy the principle of sequential rationality.) ■

In fact, for finite games of perfect information, we have the general result presented in Proposition 9.B.1.

Proposition 9.B.1: (*Zermelo's Theorem*) Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.

Proof: First, note that in finite games of perfect information, the backward induction procedure is well defined: The player who moves at each decision node has a finite number of possible choices, so optimal actions necessarily exist at each stage of the procedure (if a player is indifferent, we can choose any of her optimal actions). Moreover, the procedure fully specifies all of the players' strategies after a finite number of stages. Second, note that if no player has the same payoffs at any two terminal nodes, then the optimal actions must be *unique* at every stage of the procedure, and so in this case the backward induction procedure identifies a unique strategy profile for the game.

What remains is to show that a strategy profile identified in this way, say $\sigma = (\sigma_1, \ldots, \sigma_I)$, is necessarily a Nash equilibrium of Γ_E. Suppose that it is not. Then there is some player i who has a deviation, say to strategy $\hat{\sigma}_i$, that strictly increases her payoff given that the other players continue to play strategies σ_{-i}. That is, letting $u_i(\sigma_i, \sigma_{-i})$ be player i's payoff function,[3]

$$u_i(\hat{\sigma}_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}). \tag{9.B.1}$$

We argue that this cannot be. The proof is inductive. We shall say that decision node x has *distance* n if, among the various paths that continue from it to the terminal nodes, the maximal number of decision nodes lying between it and a terminal node is n. We let N denote the maximum distance of any decision node in the game; since Γ_E is a finite game, N is a finite number. Define $\hat{\sigma}_i(n)$ to be the strategy that plays in accordance with strategy σ_i at all nodes with distances $0, \ldots, n$, and plays in accordance with strategy $\hat{\sigma}_i$ at all nodes with distances greater than n.

By the construction of σ through the backward induction procedure, $u_i(\hat{\sigma}_i(0), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. That is, player i can do at least as well as she does with strategy $\hat{\sigma}_i$ by instead playing the moves specified in strategy σ_i at all nodes with distance 0 (i.e., at the final decision nodes in the game) and following strategy $\hat{\sigma}_i$ elsewhere.

3. To be precise, $u_i(\cdot)$ is player i's payoff function in the normal form derived from extensive form game Γ_E.

We now argue that if $u_i(\hat{\sigma}_i(n-1), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$, then $u_i(\hat{\sigma}_i(n), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. This is straightforward. The only difference between strategy $\hat{\sigma}_i(n)$ and strategy $\hat{\sigma}_i(n-1)$ is in player i's moves at nodes with distance n. In both strategies, player i plays according to σ_i at all decision nodes that follow the distance-n nodes and in accordance with strategy $\hat{\sigma}_i$ before them. But given that all players are playing in accordance with strategy profile σ after the distance-n nodes, the moves derived for the distance-n decision nodes through backward induction, namely those in σ_i, must be optimal choices for player i at these nodes. Hence, $u_i(\hat{\sigma}_i(n), \sigma_{-i}) \geq u_i(\hat{\sigma}_i(n-1), \sigma_{-i})$.

Applying induction, we therefore have $u_i(\hat{\sigma}_i(N), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. But $\hat{\sigma}_i(N) = \sigma_i$, and so we have a contradiction to (9.B.1). Strategy profile σ must therefore constitute a Nash equilibrium of Γ_E. ■

Note, incidentally, that Proposition 9.B.1 establishes the existence of a pure strategy Nash equilibrium in all finite games of perfect information.

Subgame Perfect Nash Equilibria

It is clear enough how to apply the principle of sequential rationality in Example 9.B.1 and, more generally, in finite games of perfect information. Before distilling a general solution concept, however, it is useful to discuss another example. This example suggests how we might identify Nash equilibria that satisfy the principle of sequential rationality in more general games involving imperfect information.

Example 9.B.3: We consider the same situation as in Example 9.B.1 except that firms I and E now play a simultaneous-move game after entry, each choosing either "fight" or "accommodate." The extensive and normal form representations are depicted in Figure 9.B.4.

Examining the normal form, we see that in this game there are three pure strategy Nash equilibria (σ_E, σ_I):[4]

((out, accommodate if in), (fight if firm E plays "in")),

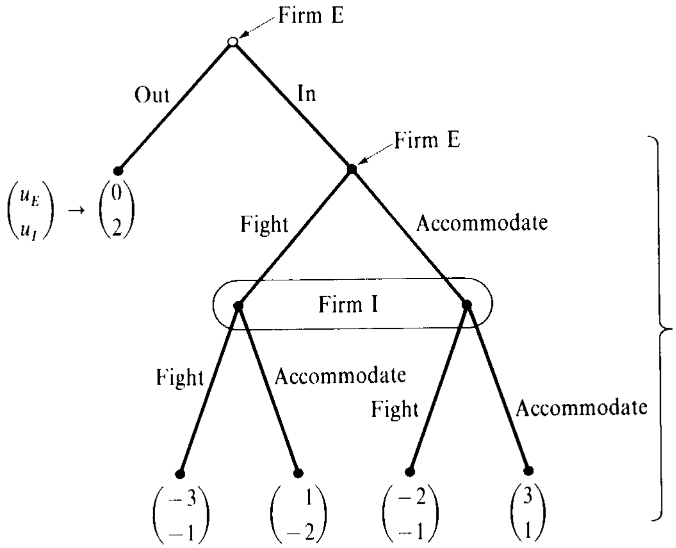
((out, fight if in), (fight if firm E plays "in")),

((in, accommodate if in), (accommodate if firm E plays "in")).

Notice, however, that (accommodate, accommodate) is the sole Nash equilibrium in the simultaneous-move game that follows entry. Thus, the firms should expect that they will both play "accommodate" following firm E's entry.[5] But if this is so, firm E

4. The entrant's strategy in the first two equilibria may appear odd. Firm E is planning to take an action conditional on entering while at the same time planning not to enter. Recall from Section 7.D, however, that a strategy is a *complete contingent plan*. Indeed, the reason we have insisted on this requirement is precisely the need to test the sequential rationality of a player's strategy.

5. Recall that throughout this chapter we maintain the assumption that rational players always play some Nash equilibrium in any strategic situation in which they find themselves (i.e., we assume that players will have mutually correct expectations). Two points about this assumption are worth noting. First, some justifications for a Nash equilibrium may be less compelling in the context of dynamic games. For example, if players never reach certain parts of a game, the stable social convention argument given in Section 8.D may no longer provide a good reason for believing that a Nash equilibrium would be played *if* that part of the game tree were reached. Second, the idea of sequential rationality can still have force even if we do not make this assumption. For example, here we would reach the same conclusion even if we assumed only that neither player would play an iteratively strictly dominated strategy in the post-entry simultaneous-move game.



A Simultaneous-Move Game:

		Firm I	
		Accommodate	Fight
Firm E	Accommodate	3, 1	−2, −1
	Fight	1, −2	−3, −1

		Firm I	
		Accommodate if E Plays "In"	Fight if E Plays "In"
Firm E	Out, Accommodate if In	0, 2	0, 2
	Out, Fight if In	0, 2	0, 2
	In, Accommodate if In	3, 1	−2, −1
	In, Fight if In	1, −2	−3, −1

Figure 9.B.4 Extensive and normal forms for Example 9.B.3. A sequentially rational Nash equilibrium must have both firms play "accommodate" after entry.

should enter. The logic of sequential rationality therefore suggests that only the last of the three equilibria is a reasonable prediction in this game. ■

The requirement of sequential rationality illustrated in this and the preceding examples is captured by the notion of a *subgame perfect Nash equilibrium* [introduced by Selten (1965)]. Before formally defining this concept, however, we need to specify what a *subgame* is.

Definition 9.B.1: A *subgame* of an extensive form game Γ_E is a subset of the game having the following properties:

(i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains *only* these nodes.

(ii) If decision node x is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node x. (That is, there are no "broken" information sets.)

Note that according to Definition 9.B.1, the game as a whole is a subgame, as may be some strict subsets of the game.[6] For example, in Figure 9.B.1, there are two subgames: the game as a whole and the part of the game tree that begins with and follows firm I's decision node. The game in Figure 9.B.4 also has two subgames: the game as a whole and the part of the game beginning with firm E's post-entry decision node. In Figure 9.B.5, the dotted lines indicate three parts of the game of Figure 9.B.4 that are *not* subgames.

Finally, note that in a finite game of perfect information, every decision node initiates a subgame. (Exercise 9.B.1 asks you to verify this fact for the game of Example 9.B.2.)

The key feature of a subgame is that, contemplated in isolation, it is a game in its own right. We can therefore apply to it the idea of Nash equilibrium predictions. In the discussion that follows, we say that a strategy profile σ in extensive form game Γ_E *induces* a Nash equilibrium in a particular subgame of Γ_E if the moves specified in σ for information sets within the subgame constitute a Nash equilibrium when this subgame is considered in isolation.

Definition 9.B.2: A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ in an I-player extensive form game Γ_E is a *subgame perfect Nash equilibrium* (SPNE) if it induces a Nash equilibrium in every subgame of Γ_E.

Note that any SPNE is a Nash equilibrium (since the game as a whole is a subgame) but that not every Nash equilibrium is subgame perfect.

Exercise 9.B.2: Consider a game Γ_E in extensive form. Argue that:

(a) If the only subgame is the game as a whole, then every Nash equilibrium is subgame perfect.

(b) A subgame perfect Nash equilibrium induces a subgame perfect Nash equilibrium in every subgame of Γ_E.

The SPNE concept isolates the reasonable Nash equilibria in Examples 9.B.1 and 9.B.3. In Example 9.B.1, any subgame perfect Nash equilibrium must have firm I playing "accommodate if firm E plays 'in'" because this is firm I's strictly dominant strategy in the subgame following entry. Likewise, in Example 9.B.3, any SPNE must have the firms both playing "accommodate" after entry because this is the unique Nash equilibrium in this subgame.

Note also that in finite games of perfect information, such as the games of Examples 9.B.1 and 9.B.2, the set of SPNEs coincides with the set of Nash equilibria that can be derived through the backward induction procedure. Recall, in particular, that in finite games of perfect information every decision node initiates a subgame. Thus, in any SPNE, the strategies must specify actions at each of the final decision nodes of the game that are optimal in the single-player subgame that begins there. Given that this must be the play at the final decision nodes in any SPNE, consider play in the subgames starting at the next-to-last decision nodes. Nash equilibrium play in these subgames, which is required in any SPNE, must have the players who

6. In the literature, the term *proper subgame* is sometimes used with the same meaning we assign to *subgame*. We choose to use the unqualified term *subgame* here to make clear that the game itself qualifies.

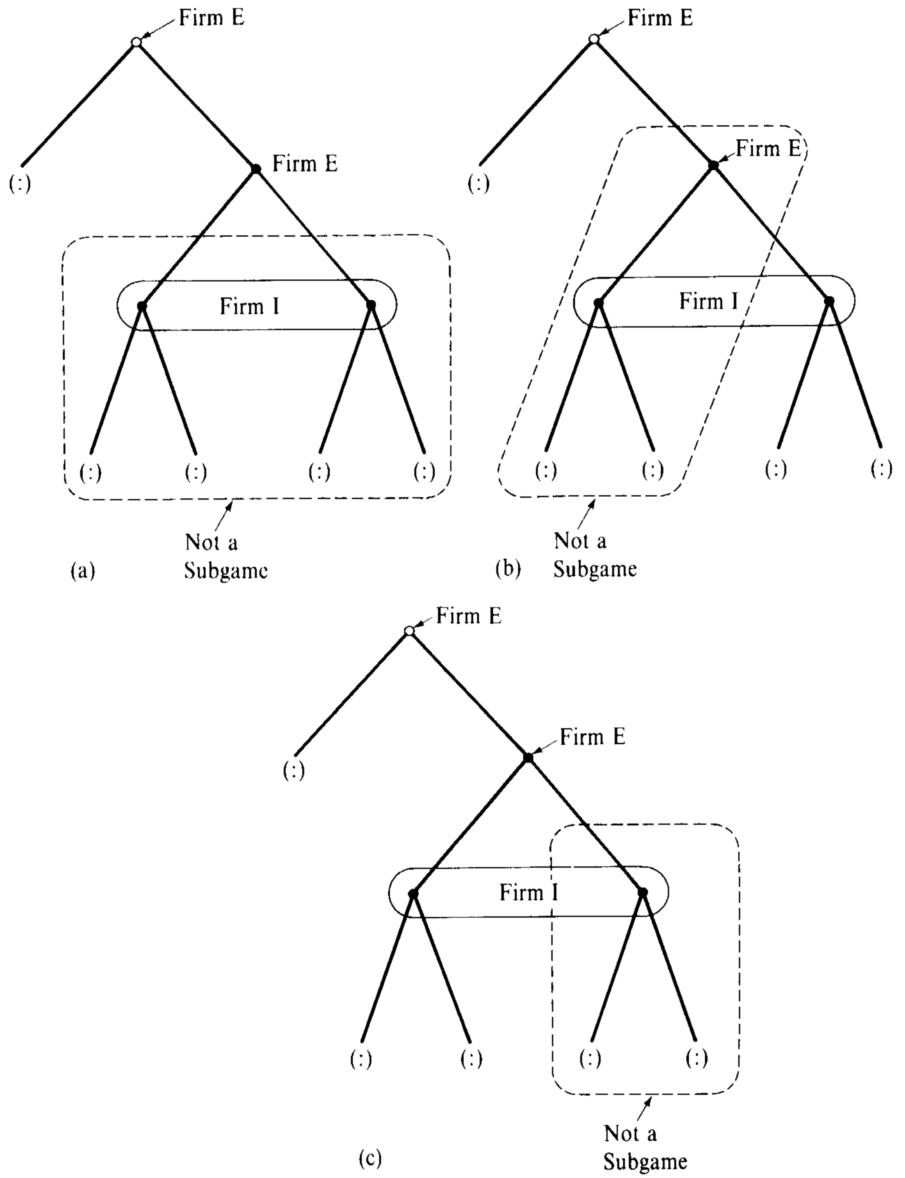


Figure 9.B.5
Three parts of the game in Figure 9.B.4 that are not subgames.

move at these next-to-last nodes choosing optimal strategies given the play that will occur at the last nodes. And so on. An implication of this fact and Proposition 9.B.1 is therefore the result stated in Proposition 9.B.2.

Proposition 9.B.2: Every finite game of perfect information Γ_E has a pure strategy subgame perfect Nash equilibrium. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.[7]

7. The result can also be seen directly from Proposition 9.B.1. Just as the strategy profile derived using the backward induction procedure constitutes a Nash equilibrium in the game as a whole, it is also a Nash equilibrium in every subgame.

In fact, to identify the set of subgame perfect Nash equilibria in a general (finite) dynamic game Γ_E, we can use a generalization of the backward induction procedure. This *generalized backward induction procedure* works as follows:

1. Start at the end of the game tree, and identify the Nash equilibria for each of the *final* subgames (i.e., those that have no other subgames nested within them).
2. Select one Nash equilibrium in each of these final subgames, and derive the reduced extensive form game in which these final subgames are replaced by the payoffs that result in these subgames when players use these equilibrium strategies.
3. Repeat steps 1 and 2 for the reduced game. Continue the procedure until every move in Γ_E is determined. This collection of moves at the various information sets of Γ_E constitutes a profile of SPNE strategies.
4. If multiple equilibria are never encountered in any step of this process, this profile of strategies is the unique SPNE. If multiple equilibria are encountered, the full set of SPNEs is identified by repeating the procedure for each possible equilibrium that could occur for the subgames in question.

The formal justification for using this generalized backward induction procedure to identify the set of SPNEs comes from the result shown in Proposition 9.B.3.

Proposition 9.B.3: Consider an extensive form game Γ_E and some subgame S of Γ_E. Suppose that strategy profile σ^S is an SPNE in subgame S, and let $\hat{\Gamma}_E$ be the reduced game formed by replacing subgame S by a terminal node with payoffs equal to those arising from play of σ^S. Then:

(i) In any SPNE σ of Γ_E in which σ^S is the play in subgame S, players' moves at information sets outside subgame S must constitute an SPNE of reduced game $\hat{\Gamma}_E$.

(ii) If $\hat{\sigma}$ is an SPNE of $\hat{\Gamma}_E$, then the strategy profile σ that specifies the moves in σ^S at information sets in subgame S and that specifies the moves in $\hat{\sigma}$ at information sets not in S is an SPNE of Γ_E.

Proof: (i) Suppose that strategy profile σ specifies play at information sets outside subgame S that does not constitute an SPNE of reduced game $\hat{\Gamma}_E$. Then there exists a subgame of $\hat{\Gamma}_E$ in which σ does not induce a Nash equilibrium. In this subgame of $\hat{\Gamma}_E$, some player has a deviation that improves her payoff, taking as given the strategies of her opponents. But then it must be that this player also has a profitable deviation in the corresponding subgame of game Γ_E. She makes the same alterations in her moves at information sets not in S and leaves her moves at information sets in S unchanged. Hence, σ could not be an SPNE of the overall game Γ_E.

(ii) Suppose that $\hat{\sigma}$ is an SPNE of reduced game $\hat{\Gamma}_E$, and let σ be the strategy in the overall game Γ_E formed by specifying the moves in σ^S at information sets in subgame S and the moves in $\hat{\sigma}$ at information sets not in S. We argue that σ induces a Nash equilibrium in every subgame of Γ_E. This follows immediately from the construction of σ for subgames of Γ_E that either lie entirely in subgame S or never intersect with subgame S (i.e., that do not have subgame S nested within them). So consider any subgame that has subgame S nested within it. If some player i has a profitable deviation in this subgame given her opponent's strategies, then she must also have a profitable deviation that leaves her moves within subgame S unchanged because, by hypothesis, a player does best within subgame S by playing the moves specified in strategy profile σ^S given that her opponents do so. But if she has such a profitable deviation, then she must have a profitable deviation in the corresponding subgame of reduced game $\hat{\Gamma}_E$, in contradiction to $\hat{\sigma}$ being an SPNE of $\hat{\Gamma}_E$. ■

Note that for the final subgames of Γ_E, the set of Nash equilibria and SPNEs coincide, because these subgames contain no nested subgames. Identifying Nash equilibria in these final subgames therefore allows us to begin the inductive application of Proposition 9.B.3.

This generalized backward induction procedure reduces to our previous backward induction procedure in the case of games of perfect information. But it also applies to games of imperfect information. Example 9.B.3 provides a simple illustration. There we can identify the unique SPNE by first identifying the unique Nash equilibrium in the post-entry subgame: (accommodate, accommodate). Having done this, we can replace this subgame with the payoffs that result from equilibrium play in it. The reduced game that results is then much the same as that shown in Figure 9.B.2, the only difference being that firm E's payoff from playing "in" is now 3 instead of 2. Hence, in this manner, we can derive the unique SPNE of Example 9.B.3: $(\sigma_E, \sigma_I) =$ ((in, accommodate if in), (accommodate if firm E plays "in")).

The game in Example 9.B.3 is simple to solve in two respects. First, there is a unique equilibrium in the post-entry subgame. If this were not so, behavior earlier in the game could depend on *which* equilibrium resulted after entry. Example 9.B.4 illustrates this point:[8]

Example 9.B.4: *The Niche Choice Game.* Consider a modification of Example 9.B.3 in which instead of having the two firms choose whether to fight or accommodate each other, we suppose that there are actually two niches in the market, one large and one small. After entry, the two firms decide simultaneously which niche they will be in. For example, the niches might correspond to two types of customers, and the firms may be deciding to which type they are targeting their product design. Both firms lose money if they choose the same niche, with more lost if it is the small niche. If they choose different niches, the firm that targets the large niche earns a profit, and the firm with the small niche incurs a loss, but a smaller loss than if the two firms targeted the same niche. The extensive form of this game is depicted in Figure 9.B.6.

To determine the SPNE of this game, consider the post-entry subgame first. There are two pure strategy Nash equilibria of this simultaneous-move game: (large niche, small niche) and (small niche, large niche).[9] In any pure strategy SPNE, the firms' strategies must induce one of these two Nash equilibria in the post-entry subgame. Suppose, first, that the firms will play (large niche, small niche). In this case, the payoffs from reaching the post-entry subgame are $(u_E, u_I) = (1, -1)$, and the reduced game is as depicted in Figure 9.B.7(a). The entrant optimally chooses to enter in this

8. Similar issues can arise in games of perfect information when a player is indifferent between two actions. However, the presence of multiple equilibria in subgames involving simultaneous play is, in a sense, a more robust phenomenon. Multiple equilibria are generally robust to small changes in players' payoffs, but ties in games of perfect information are not.

9. We restrict attention here to pure strategy SPNEs. There is also a mixed strategy Nash equilibrium in the post-entry subgame. Exercise 9.B.6 asks you to investigate the implications of this mixed strategy play being the post-entry equilibrium behavior.

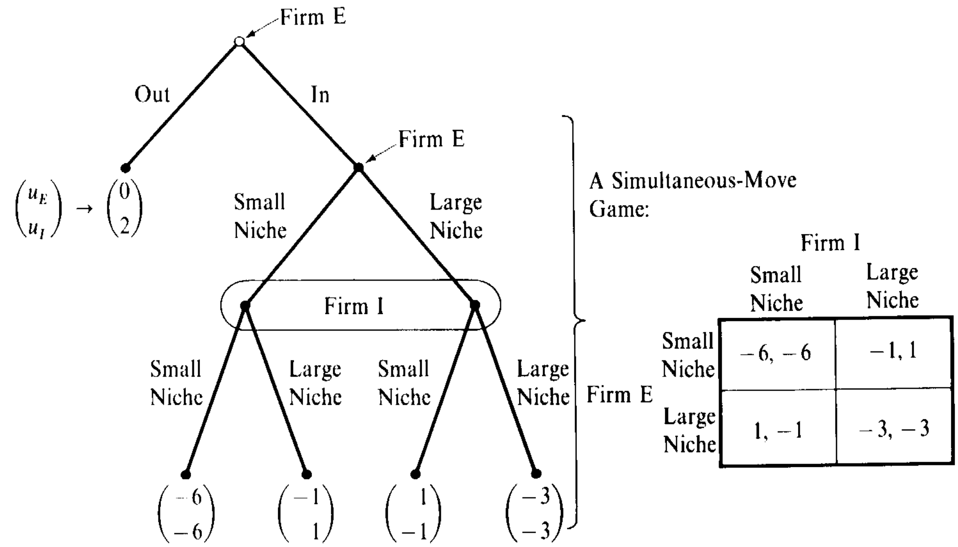
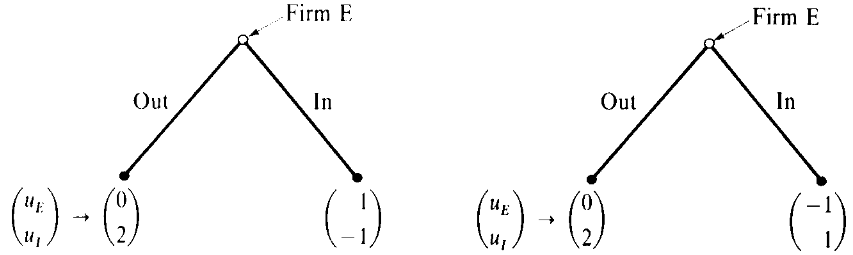


Figure 9.B.6 Extensive form for the Niche Choice game. The post-entry subgame has multiple Nash equilibria.



(a) (b)

Figure 9.B.7 Reduced games after identifying (pure strategy) Nash equilibria in the post-entry subgame of the Niche Choice game. (a) Reduced game if (large niche, small niche) is post-entry equilibrium. (b) Reduced game if (small niche, large niche) is post-entry equilibrium.

case. Hence, one SPNE is (σ_E, σ_I) = ((in, large niche if in), (small niche if firm E plays "in")).

Now suppose that the post-entry play is (small niche, large niche). Then the payoffs from reaching the post-entry game are $(u_E, u_I) = (-1, 1)$, and the reduced game is that depicted in Figure 9.B.7(b). The entrant optimally chooses not to enter in this case. Hence, there is a second pure strategy SPNE: (σ_E, σ_I) = ((out, small niche if in), (large niche if firm E plays "in")). ■

A second sense in which the game in Example 9.B.3 is simple to solve is that it involves only one subgame other than the game as a whole. Like games of perfect information, a game with imperfect information may in general have *many* subgames, with one subgame nested within another, and that larger subgame nested within a still larger one, and so on.

One interesting class of imperfect information games in which the generalized backward induction procedure gives a very clean conclusion is described in Proposition 9.B.4.

Proposition 9.B.4: Consider an I-player extensive form game Γ_E involving successive play of T simultaneous-move games, $\Gamma_N^t = [I, \{\Delta(S_i^t)\}, \{u_i^t(\cdot)\}]$ for $t = 1, \ldots, T$, with the players observing the pure strategies played in each game immediately after its play is concluded. Assume that each player's payoff is equal to the sum of her payoffs in the plays of the T games. If there is a unique Nash equilibrium in each game Γ_N^t, say $\sigma^t = (\sigma_1^t, \ldots, \sigma_I^t)$, then there is a unique SPNE of Γ_E and it consists of each player i playing strategy σ_i^t in each game Γ_N^t regardless of what has happened previously.

Proof: The proof is by induction. The result is clearly true for $T = 1$. Now suppose it is true for all $T \leq n - 1$. We will show that it is true for $T = n$.

We know by hypothesis that in any SPNE of the overall game, after play of game Γ_N^1 the play in the remaining $n - 1$ simultaneous-move games must simply involve play of the Nash equilibrium of each game (since any SPNE of the overall game induces an SPNE in each of its subgames). Let player i earn G_i from this equilibrium play in these $n - 1$ games. Then in the reduced game that replaces all the subgames that follow Γ_N^1 with their equilibrium payoffs, player i earns an overall payoff of $u_i(s_1^1, \ldots, s_I^1) + G_i$ if $(s_1^1, \ldots, s_I^1)$ is the profile of pure strategies played in game Γ_N^1. The unique Nash equilibrium of this reduced game is clearly σ^1. Hence, the result is also true for $T = n$. ■

The basic idea behind Proposition 9.B.4 is an application of backward induction logic: Play in the last game must result in the unique Nash equilibrium of that game being played because at that point players essentially face just that game. But if play in the last game is predetermined, then when players play the next-to-last game, it is again as if they were playing just *that* game in isolation (think of the case where $T = 2$). And so on.

An interesting aspect of Proposition 9.B.4 is the way the SPNE concept rules out history dependence of strategies in the class of games considered there. In general, a player's strategy could potentially promise later rewards or punishments to other players if they take particular actions early in the game. But as long as each of the component games has a unique Nash equilibrium, SPNE strategies cannot be history dependent.[10]

Exercises 9.B.9 to 9.B.11 provide some additional examples of the use of the subgame perfect Nash equilibrium concept. In Appendix A we also study an important economic application of subgame perfection to a finite game of perfect information (albeit one with an infinite number of possible moves at some decision nodes): a finite horizon model of bilateral bargaining.

Up to this point, our analysis has assumed that the game being studied is finite. This has been important because it has allowed us to identify subgame perfect Nash equilibria by starting at the end of the game and working backward. As a general matter, in games in which there can be an infinite sequence of moves (so that some paths through the tree never reach a terminal node), the definition of a subgame perfect Nash equilibrium remains that given in Definition 9.B.2: Strategies must induce a Nash equilibrium in every subgame. Nevertheless, the lack of a definite finite point of termination of the game can reduce the power of the SPNE concept because we can no longer use the end of the game to pin down behavior. In games in which there is always a future, a wide range of behaviors can sometimes be justified as sequentially rational (i.e., as part of an SPNE). A striking example of this sort arises in

10. This lack of history dependence depends importantly on the uniqueness assumption of Proposition 9.B.4. With multiple Nash equilibria in the component games, we can get outcomes that are not merely the repeated play of the static Nash equilibria. (See Exercise 9.B.9 for an example.)

Chapter 12 and its Appendix A when we consider *infinitely repeated games* in the context of studying oligopolistic pricing.

Nevertheless, it is not always the case that an infinite horizon weakens the power of the subgame perfection criterion. In Appendix A of this chapter, we study an infinite horizon model of bilateral bargaining in which the SPNE concept predicts a unique outcome, and this outcome coincides with the limiting outcome of the corresponding finite horizon bargaining model as the horizon grows long.

The methods used to identify subgame perfect Nash equilibria in infinite horizon games are varied. Sometimes, the method involves showing that the game can effectively be truncated because after a certain point it is obvious what equilibrium play must be (see Exercise 9.B.11). In other situations, the game possesses a stationarity property that can be exploited; the analysis of the infinite horizon bilateral bargaining model in Appendix A is one example of this kind.

After the preceding analysis, the logic of sequential rationality may seem unassailable. But things are not quite so clear. For example, nowhere could the principle of sequential rationality seem on more secure footing than in finite games of perfect information. But chess is a game of this type (the game ends if 50 moves occur without a piece being taken or a pawn being moved), and so its "solution" should be simple to predict. Of course, it is exactly players' *inability* to do so that makes it an exciting game to play. The same could be said even of the much simpler game of Chinese checkers. It is clear that in practice, players may be only boundedly rational. As a result, we might feel more comfortable with our rationality hypotheses in games that are relatively simple, in games where repetition helps players learn to think through the game, or in games where large stakes encourage players to do so as much as possible. Of course, the possibility of bounded rationality is not a concern limited to dynamic games and subgame perfect Nash equilibria; it is also relevant for simultaneous-move games containing many possible strategies.

There is, however, an interesting tension present in the SPNE concept that is related to this bounded rationality issue and that does not arise in the context of simultaneous-move games. In particular, the SPNE concept insists that players should play an SPNE wherever they find themselves in the game tree, even after a sequence of events that is contrary to the predictions of the theory. To see this point starkly, consider the following example due to Rosenthal (1981), known as the *Centipede game*.

Example 9.B.5: *The Centipede Game.* In this finite game of perfect information, there are two players, 1 and 2. The players each start with 1 dollar in front of them. They alternate saying "stop" or "continue," starting with player 1. When a player says "continue," 1 dollar is taken by a referee from her pile and 2 dollars are put in her opponent's pile. As soon as either player says "stop," play is terminated, and each player receives the money currently in her pile. Alternatively, play stops if both players' piles reach 100 dollars. The extensive form for this game is depicted in Figure 9.B.8.

The unique SPNE in this game has both players saying "stop" whenever it is their turn, and the players each receive 1 dollar in this equilibrium. To see this, consider player 2's move at the final decision node of the game (after the players have said "continue" a total of 197 times). Her optimal move if play reaches this point is to say "stop"; by doing so, she receives 101 dollars compared with a payoff of 100 dollars if she says "continue." Now consider what happens if play reaches the next-to-last decision node. Player 1, anticipating player 2's move at the final decision node, also says "stop"; doing so, she earns 99 dollars, compared with 98 dollars if she says "continue." Continuing backward through the tree in this fashion, we identify saying "stop" as the optimal move at every decision node.

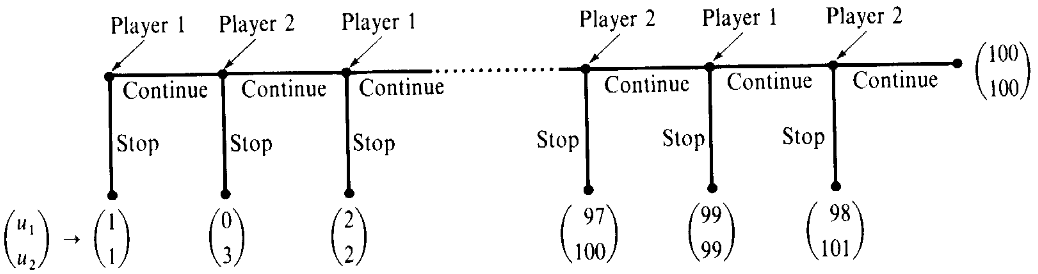


Figure 9.B.8 The Centipede game.

A striking aspect of the SPNE in the Centipede game is how bad it is for the players. They each get 1 dollar, whereas they might get 100 dollars by repeatedly saying "continue."

Is this (unique) SPNE in the Centipede game a reasonable prediction? Consider player 1's initial decision to say "stop." For this to be rational, player 1 must be pretty sure that if instead she says "continue," player 2 will say "stop" at her first turn. Indeed, "continue" would be better for player 1 as long as she could be sure that player 2 would say "continue" at her next move. Why might player 2 respond to player 1 saying "continue" by also saying "continue"? First, as we have pointed out, player 2 might not be fully rational, and so she might not have done the backward induction computation assumed in the SPNE concept. More interestingly, however, once she sees that player 1 has chosen "continue"—an event that should never happen according to the SPNE prediction—she might entertain the possibility that player 1 is not rational in the sense demanded by the SPNE concept. If, as a result, she thinks that player 1 would say "continue" at her next move if given the chance, then player 2 would want to say "continue" herself. The SPNE concept denies this possibility, instead assuming that at any point in the game, players will assume that the remaining play of the game will be an SPNE even if play up to that point has contradicted the theory. One way of resolving this tension is to view the SPNE theory as treating any deviation from prescribed play as the result of an extremely unlikely "mistake" that is unlikely to occur again. In Appendix B, we discuss one concept that makes this idea explicit. ■

9.C Beliefs and Sequential Rationality

Although subgame perfection is often very useful in capturing the principle of sequential rationality, sometimes it is not enough. Consider Example 9.C.1's adaptation of the entry game studied in Example 9.B.1.

Example 9.C.1: We now suppose that there are two strategies firm E can use to enter, "in_1" and "in_2," and that the incumbent is unable to tell which strategy firm E has used if entry occurs. Figure 9.C.1 depicts this game and its payoffs.

As in the original entry game in Example 9.B.1, there are two pure strategy Nash equilibria here: (out, fight if entry occurs) and (in_1, accommodate if entry occurs). Once again, however, the first of these does not seem very reasonable; regardless of what entry strategy firm E has used, the incumbent prefers to accommodate once entry has occurred. *But the criterion of subgame perfection is of absolutely no use here*: Because the only subgame is the game as a whole, both pure strategy Nash equilibria are subgame perfect. ■

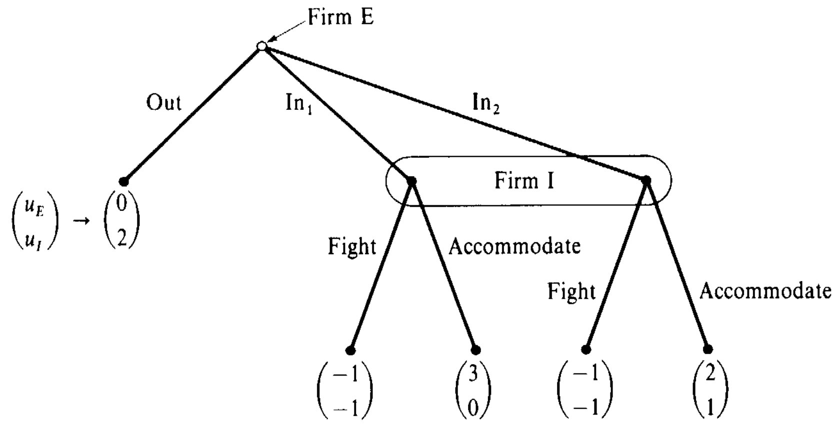


Figure 9.C.1 Extensive form for Example 9.C.1. The SPNE concept may fail to insure sequential rationality.

How can we eliminate the unreasonable equilibrium here? One possibility, which is in the spirit of the principle of sequential rationality, might be to insist that the incumbent's action after entry be optimal for *some belief* that she might have about which entry strategy was used by firm E. Indeed, in Example 9.C.1, "fight if entry occurs" is not an optimal choice for *any* belief that firm I might have. This suggests that we may be able to make some progress by formally considering players' beliefs and using them to test the sequential rationality of players' strategies.

We now introduce a solution concept, which we call a *weak perfect Bayesian equilibrium* [Myerson (1991) refers to this same concept as a *weak sequential equilibrium*], that extends the principle of sequential rationality by formally introducing the notion of beliefs.[11] It requires, roughly, that at any point in the game, a player's strategy prescribe optimal actions from that point on given her opponents' strategies and her beliefs about what has happened so far in the game and that her beliefs be consistent with the strategies being played.

To express this notion formally, we must first formally define the two concepts that are its critical components: the notions of a *system of beliefs* and the *sequential rationality of strategies*. Beliefs are simple.

Definition 9.C.1: A *system of beliefs* μ in extensive form game Γ_E is a specification of a probability $\mu(x) \in [0, 1]$ for each decision node x in Γ_E such that

$$\textstyle\sum_{x \in H} \mu(x) = 1$$

for all information sets H.

A system of beliefs can be thought of as specifying, for each information set, a probabilistic assessment by the player who moves at that set of the relative likelihoods of being at each of the information set's various decision nodes, conditional upon play having reached that information set.

11. The concept of a *perfect Bayesian equilibrium* was first developed to capture the requirements of sequential rationality in dynamic games with incomplete information, that is (using the terminology introduced in Section 8.E), in dynamic Bayesian games. The *weak perfect Bayesian equilibrium* concept is a variant that is introduced here primarily for pedagogic purposes (the reason for the modifier *weak* will be made clear later in this section). Myerson (1991) refers to this same concept as a *weak sequential equilibrium* because it may also be considered a weak variant of the *sequential equilibrium* concept introduced in Definition 9.C.4.

To define sequential rationality, it is useful to let $E[u_i \mid H, \mu, \sigma_i, \sigma_{-i}]$ denote player i's expected utility starting at her information set H if her beliefs regarding the conditional probabilities of being at the various nodes in H are given by μ, if she follows strategy σ_i, and if her rivals use strategies σ_{-i}. [We will not write out the formula for this expression explicitly, although it is conceptually straightforward: Pretend that the probability distribution $\mu(x)$ over nodes $x \in H$ is generated by nature; then player i's expected payoff is determined by the probability distribution that is induced on the terminal nodes by the combination of this initial distribution plus the players' strategies from this point on.]

Definition 9.C.2: A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ in extensive form game Γ_E is *sequentially rational at information set H given a system of beliefs* μ if, denoting by $\iota(H)$ the player who moves at information set H, we have

$$E[u_{\iota(H)} \mid H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq E[u_{\iota(H)} \mid H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\tilde{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile σ satisfies this condition for *all* information sets H, then we say that σ is *sequentially rational given belief system* μ.

In words, a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ is sequentially rational if no player finds it worthwhile, once one of her information sets has been reached, to revise her strategy given her beliefs about what has already occurred (as embodied in μ) and her rivals' strategies.

With these two notions, we can now define a weak perfect Bayesian equilibrium. The definition involves two conditions: First, strategies must be sequentially rational given beliefs. Second, whenever possible, beliefs must be consistent with the strategies. The idea behind the consistency condition on beliefs is much the same as the idea underlying the concept of Nash equilibrium (see Section 8.D): In an equilibrium, players should have correct beliefs about their opponents' strategy choices.

To motivate the specific consistency requirement on beliefs to be made in the definition of a weak perfect Bayesian equilibrium, consider how we might define the notion of consistent beliefs in the special case in which each player's equilibrium strategy assigns a strictly positive probability to each possible action at every one of her information sets (known as a *completely mixed strategy*).[12] In this case, every information set in the game is reached with positive probability. The natural notion of beliefs being consistent with the play of the equilibrium strategy profile σ is in this case straightforward: For each node x in a given player's information set H, the player should compute the probability of reaching that node given play of strategies σ, Prob $(x \mid \sigma)$, and she should then assign conditional probabilities of being at each of these nodes given that play has reached this information set using *Bayes' rule*:[13]

$$\text{Prob}\,(x \mid H, \sigma) = \frac{\text{Prob}\,(x \mid \sigma)}{\sum_{x' \in H} \text{Prob}\,(x' \mid \sigma)}.$$

12. Equivalently, a completely mixed strategy can be thought of as a strategy that assigns a strictly positive probability to each of the player's pure strategies in the normal form derived from extensive form game Γ_E.

13. Bayes' rule is a basic principle of statistical inference. See, for example, DeGroot (1970), where it is referred to as *Bayes' theorem*.

As a concrete example, suppose that in the game in Example 9.C.1, firm E is using the completely mixed strategy that assigns a probability of $\frac{1}{4}$ to "out," $\frac{1}{2}$ to "in$_1$," and $\frac{1}{4}$ to "in$_2$." Then the probability of reaching firm I's information set given this strategy is $\frac{3}{4}$. Using Bayes' rule, the probability of being at the left node of firm I's information set conditional on this information set having been reached is $\frac{2}{3}$, and the conditional probability of being at the right node in the set is $\frac{1}{3}$. For firm I's beliefs following entry to be consistent with firm E's strategy, firm I's beliefs should assign exactly these probabilities.

The more difficult issue arises when players are not using completely mixed strategies. In this case, some information sets may no longer be reached with positive probability, and so we cannot use Bayes' rule to compute conditional probabilities for the nodes in these information sets. At an intuitive level, this problem corresponds to the idea that even if players were to play the game repeatedly, the equilibrium play would generate no experience on which they could base their beliefs at these information sets. The weak perfect Bayesian equilibrium concept takes an agnostic view toward what players should believe if play were to reach these information sets unexpectedly. In particular, it allows us to assign *any* beliefs at these information sets. It is in this sense that the modifier *weak* is appropriately attached to this concept.

We can now give a formal definition.

Definition 9.C.3: A profile of strategies and system of beliefs (σ, μ) is a *weak perfect Bayesian equilibrium* (weak PBE) in extensive form game Γ_E if it has the following properties:

(i) The strategy profile σ is sequentially rational given belief system μ.

(ii) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible. That is, for any information set H such that Prob $(H \mid \sigma) > 0$ (read as "the probability of reaching information set H is positive under strategies σ"), we must have

$$\mu(x) = \frac{\text{Prob}\,(x \mid \sigma)}{\text{Prob}\,(H \mid \sigma)} \quad \text{for all } x \in H.$$

It should be noted that the definition formally incorporates beliefs as part of an equilibrium by identifying a *strategy–beliefs pair* (σ, μ) as a weak perfect Bayesian equilibrium. In the literature, however, it is not uncommon to see this treated a bit loosely: a set of strategies σ will be referred to as an equilibrium with the meaning that there is at least one associated set of beliefs μ such that (σ, μ) satisfies Definition 9.C.3. At times, however, it can be very useful to be more explicit about what these beliefs are, such as when testing them against some of the "reasonableness" criteria that we discuss in Section 9.D.

A useful way to understand the relationship between the weak PBE concept and that of Nash equilibrium comes in the characterization of Nash equilibrium given in Proposition 9.C.1.

Proposition 9.C.1: A strategy profile σ is a Nash equilibrium of extensive form game Γ_E if and only if there exists a system of beliefs μ such that

(i) The strategy profile σ is sequentially rational given belief system μ *at all information sets H such that Prob* $(H \mid \sigma) > 0$.

(ii) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible.

Exercise 9.C.1 asks you to prove this result. The italicized portion of condition (i) is the only change from Definition 9.C.3: For a Nash equilibrium, we require sequential rationality only on the equilibrium path. Hence, a weak perfect Bayesian equilibrium of game Γ_E is a Nash equilibrium, but not every Nash equilibrium is a weak PBE.

We now illustrate the application of the weak PBE concept in several examples. We first consider how the concept performs in Example 9.C.1.

Example 9.C.1 Continued: Clearly, firm I must play "accommodate if entry occurs" in any weak perfect Bayesian equilibrium because that is firm I's optimal action starting at its information set for *any* system of beliefs. Thus, the Nash equilibrium strategies (out, fight if entry occurs) cannot be part of any weak PBE.

What about the other pure strategy Nash equilibrium, (in_1, accommodate if entry occurs)? To show that this strategy profile *is* part of a weak PBE, we need to supplement these strategies with a system of beliefs that satisfy criterion (ii) of Definition 9.C.3 and that lead these strategies to be sequentially rational. Note first that to satisfy criterion (ii), the incumbent's beliefs must assign probability 1 to being at the left node in her information set because this information set is reached with positive probability given the strategies (in_1, accommodate if entry occurs) [a specification of beliefs at this information set fully describes a system of beliefs in this game because the only other information set is a singleton]. Moreover, these strategies are, indeed, sequentially rational given this system of beliefs. In fact, this strategy–beliefs pair is the unique weak PBE in this game (pure or mixed). ■

Examples 9.C.2 and 9.C.3 provide further illustrations of the application of the weak PBE concept.

Example 9.C.2: Consider the following "joint venture" entry game: Now there is a second potential entrant E2. The story is as follows: Firm E1 has the essential capability to enter the market but lacks some important capability that firm E2 has. As a result, E1 is considering proposing a joint venture with E2 in which E2 shares its capability with E1 and the two firms split the profits from entry. Firm E1 has three initial choices: enter directly on its own, propose a joint venture with E2, or stay out of the market. If it proposes a joint venture, firm E2 can either accept or decline. If E2 accepts, then E1 enters with E2's assistance. If not, then E1 must decide whether to enter on its own. The incumbent can observe whether E1 has entered, but not whether it is with E2's assistance. Fighting is the best response for the incumbent if E1 is unassisted (E1 can then be wiped out quickly) but is not optimal for the incumbent if E1 is assisted (E1 is then a tougher competitor). Finally, if E1 is unassisted, it wants to enter only if the incumbent accommodates; but if E1 is assisted by E2, then because it will be such a strong competitor, its entry is profitable regardless of whether the incumbent fights. The extensive form of this game is depicted in Figure 9.C.2.

To identify the weak PBE of this game note first that, in any weak PBE, firm E2 must accept the joint venture if firm E1 proposes it because E2 is thereby assured of a positive payoff regardless of firm I's strategy. But if so, then in any weak PBE

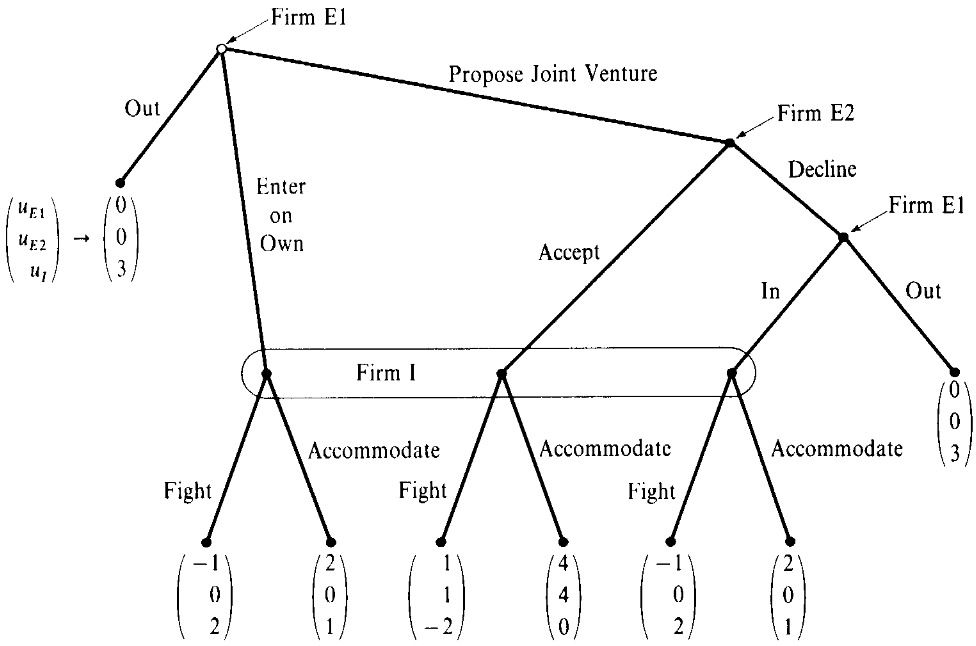


Figure 9.C.2 Extensive form for Example 9.C.2.

firm E1 must propose the joint venture since if firm E2 will accept its proposal, then firm E1 does better proposing the joint venture than it does by either staying out or entering on its own, regardless of firm I's post-entry strategy. Next, these two conclusions imply that firm I's information set is reached with positive probability (in fact, with certainty) in any weak PBE. Applying Bayesian updating at this information set, we conclude that the beliefs at this information set must assign a probability of 1 to being at the middle node. Given this, in any weak PBE firm I's strategy must be "accommodate if entry occurs." Finally, if firm I is playing "accommodate if entry occurs," then firm E1 must enter if it proposes a joint venture that firm E2 then rejects.

We conclude that the unique weak PBE in this game is a strategy–beliefs pair with strategies of $(\sigma_{E1}, \sigma_{E2}, \sigma_I)$ = ((propose joint venture, in if E2 declines), (accept), (accommodate if entry occurs)) and a belief system of μ (middle node of incumbent's information set) = 1. Note that this is not the only Nash equilibrium or, for that matter, the only SPNE. For example, $(\sigma_{E1}, \sigma_{E2}, \sigma_I)$ = ((out, out if E2 declines), (decline), (fight if entry occurs)) is an SPNE in this game. ■

Example 9.C.3: In the games of Examples 9.C.1 and 9.C.2 the trick to identifying the weak PBEs consisted of seeing that some player had an optimal strategy that was independent of her beliefs and/or the future play of her opponents. In the game depicted in Figure 9.C.3, however, this is not so for either player. Firm I is now willing to fight if she thinks that firm E has played "in$_1$," and the optimal strategy for firm E depends on firm I's behavior (note that $\gamma > -1$).

To solve this game, we look for a *fixed point* at which the behavior generated by beliefs is consistent with these beliefs. We restrict attention to the case where $\gamma > 0$. [Exercise 9.C.2 asks you to determine the set of weak PBEs when $\gamma \in (-1, 0)$.] Let σ_F be the probability that firm I fights after entry, let μ_1 be firm I's belief that

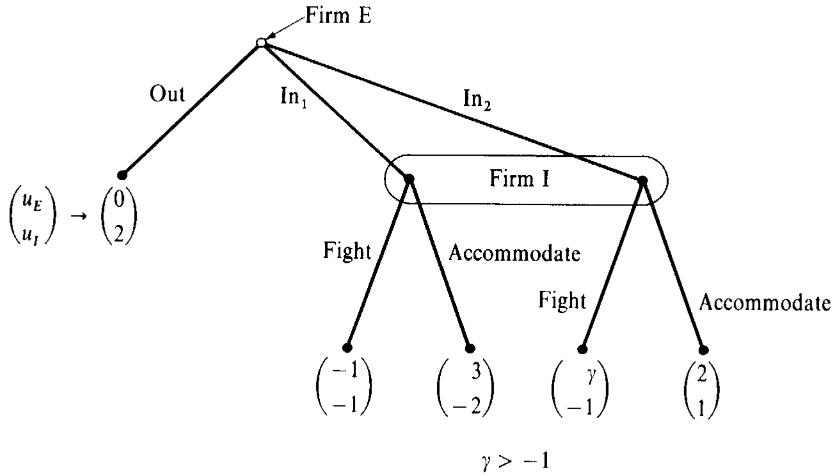


Figure 9.C.3
Extensive form for Example 9.C.3.

"in_1" was E's entry strategy if entry has occurred, and let $\sigma_0, \sigma_1, \sigma_2$ denote the probabilities with which firm E actually chooses "out," "in_1," and "in_2," respectively.

Note, first, that firm I is willing to play "fight" with positive probability if and only if $-1 \geq -2\mu_1 + 1(1 - \mu_1)$, or $\mu_1 \geq \frac{2}{3}$.

Suppose, first, that $\mu_1 > \frac{2}{3}$ in a weak PBE. Then firm I must be playing "fight" with probability 1. But then firm E must be playing "in_2" with probability 1 (since $\gamma > 0$), and the weak PBE concept would then require that $\mu_1 = 0$, which is a contradiction.

Suppose, instead, that $\mu_1 < \frac{2}{3}$ in a weak PBE. Then firm I must be playing "accommodate" with probability 1. But, if so, then firm E must be playing "in_1" with probability 1, and the weak PBE concept then requires that $\mu_1 = 1$, another contradiction.

Hence, in any weak PBE of this game, we must have $\mu_1 = \frac{2}{3}$. If so, then firm E must be randomizing in the equilibrium with positive probabilities attached to both "in_1" and "in_2" and with "in_1" twice as likely as "in_2." This means that firm I's probability of playing "fight" must make firm E indifferent between "in_1" and "in_2." Hence, we must have $-1\sigma_F + 3(1 - \sigma_F) = \gamma\sigma_F + 2(1 - \sigma_F)$, or $\sigma_F = 1/(\gamma + 2)$. Firm E's payoff from playing "in_1" or "in_2" is then $(3\gamma + 2)/(\gamma + 2) > 0$, and so firm E must play "out" with zero probability. Therefore, the unique weak PBE in this game when $\gamma > 0$ has $(\sigma_0, \sigma_1, \sigma_2) = (0, \frac{2}{3}, \frac{1}{3})$, $\sigma_F = 1/(\gamma + 2)$, and $\mu_1 = \frac{2}{3}$. ■

Strengthenings of the Weak Perfect Bayesian Equilibrium Concept

We have referred to the concept defined in Definition 9.C.3 as a *weak* perfect Bayesian equilibrium because the consistency requirements that it puts on beliefs are very minimal: The *only* requirement for beliefs, other than that they specify nonnegative probabilities which add to 1 within each information set, is that they are consistent with the equilibrium strategies on the equilibrium path, in the sense of being derived from them through Bayes' rule. *No restrictions at all are placed on beliefs off the equilibrium path* (i.e., at information sets not reached with positive probability with play of the equilibrium strategies). In the literature, a number of strengthenings of this concept that put additional consistency restrictions on off-the-equilibrium-path

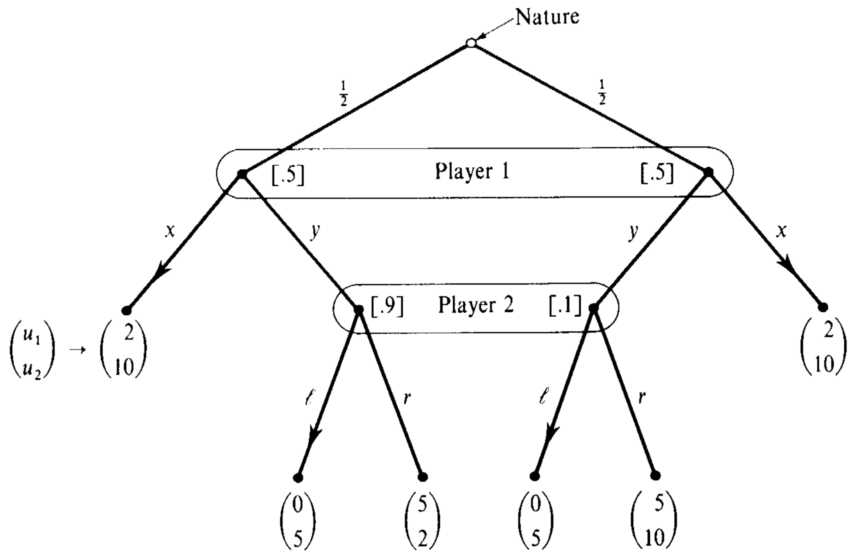


Figure 9.C.4 Extensive form for Example 9.C.4. Beliefs in a weak PBE may not be structurally consistent.

beliefs are used. Examples 9.C.4 and 9.C.5 illustrate why a strengthening of the weak PBE concept is often needed.

Example 9.C.4: Consider the game shown in Figure 9.C.4. The pure strategies and beliefs depicted in the figure constitute a weak PBE (the strategies are indicated by arrows on the chosen branches at each information set, and beliefs are indicated by numbers in brackets at the nodes in the information sets). The beliefs satisfy criterion (ii) of Definition 9.C.3; only player 1's information set is reached with positive probability, and player 1's beliefs there do reflect the probabilities assigned by nature. But the beliefs specified for player 2 in this equilibrium are not very sensible; player 2's information set can be reached only if player 1 deviates by instead choosing action y with positive probability, a deviation that must be independent of nature's actual move, since player 1 is ignorant of it. Hence, player 2 could reasonably have only beliefs that assign an equal probability to the two nodes in her information set. Here we see that it is desirable to require that beliefs at least be "structurally consistent" off the equilibrium path in the sense that there is *some* subjective probability distribution over strategy profiles that could generate probabilities consistent with the beliefs. ■

Example 9.C.5: A second and more significant problem is that a weak perfect Bayesian equilibrium need not be subgame perfect. To see this, consider again the entry game in Example 9.B.3. One weak PBE of this game involves strategies of (σ_E, σ_I) = ((out, accommodate if in), (fight if firm E plays "in")) combined with beliefs for firm I that assign probability 1 to firm E having played "fight." This weak PBE is shown in Figure 9.C.5. But note that these strategies are not subgame perfect; they do not specify a Nash equilibrium in the post-entry subgame.

The problem is that firm I's post-entry belief about firm E's post-entry play is unrestricted by the weak PBE concept because firm I's information set is off the equilibrium path. ■

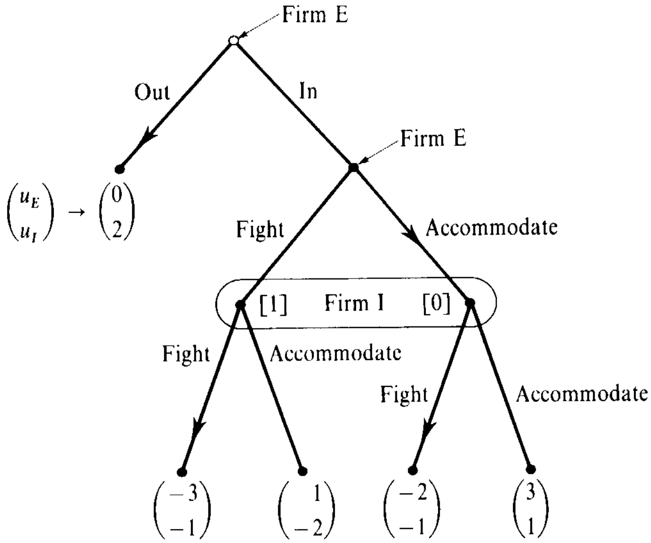


Figure 9.C.5
Extensive form for Example 9.C.5. A weak PBE may not be subgame perfect.

These two examples indicate that the weak PBE concept can be too weak. Thus, in applications in the literature, extra consistency restrictions on beliefs are often added to the weak PBE concept to avoid these problems, with the resulting solution concept referred to as a *perfect Bayesian equilibrium.* (As a simple example, restricting attention to equilibria that induce a weak PBE in every subgame insures subgame perfection.) We shall also do this when necessary later in the book; see, in particular, the discussion of signaling in Section 13.C. For formal definitions and discussion of some notions of perfect Bayesian equilibrium, see Fudenberg and Tirole (1991a) and (1991b).

An important closely related equilibrium notion that also strengthens the weak PBE concept by embodying additional consistency restrictions on beliefs is the *sequential equilibrium* concept developed by Kreps and Wilson (1982). In contrast to notions of perfect Bayesian equilibrium (such as the one we develop in Section 13.C), the sequential equilibrium concept introduces these consistency restrictions indirectly through the formalism of a limiting sequence of strategies. Definition 9.C.4 describes its requirements.

Definition 9.C.4: A strategy profile and system of beliefs (σ, μ) is a *sequential equilibrium* of extensive form game Γ_E if it has the following properties:

(i) Strategy profile σ is sequentially rational given belief system μ.

(ii) There exists a sequence of completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$, with $\lim_{k\to\infty} \sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty} \mu^k$, where μ^k denotes the beliefs derived from strategy profile σ^k using Bayes' rule.

In essence, the sequential equilibrium notion requires that beliefs be justifiable as coming from some set of totally mixed strategies that are "close to" the equilibrium strategies σ (i.e., a small perturbation of the equilibrium strategies). This can be viewed as requiring that players can (approximately) justify their beliefs by some story in which, with some small probability, players make mistakes in choosing their strategies. Note that every sequential equilibrium is a weak perfect Bayesian equilibrium because the limiting beliefs in Definition 9.C.4 exactly coincide with the beliefs derived from the equilibrium strategies σ via Bayes' rule on the outcome path of strategy profile σ. But, in general, the reverse is not true.

As we now show, the sequential equilibrium concept strengthens the weak perfect Bayesian equilibrium concept in a manner that avoids the problems identified in Examples 9.C.4 and 9.C.5.

Example 9.C.4 Continued: Consider again the game in Figure 9.C.4. In this game, all beliefs that can be derived from any sequence of totally mixed strategies assign equal probability to the two nodes in player 2's information set. Given this fact, in any sequential equilibrium player 2 must play r and player 1 must therefore play y. In fact, strategies (y, r) and beliefs giving equal probability to the two nodes in both players' information sets constitute the unique sequential equilibrium of this game. ■

Example 9.C.5 Continued: The unique sequential equilibrium strategies in the game in Example 9.C.5 (see Figure 9.C.5) are those of the unique SPNE: ((in, accommodate if in), (accommodate if firm E plays "in")). To verify this point, consider any totally mixed strategy $\bar{\sigma}$ and any node x in firm I's information set, which we denote by H_I. Letting z denote firm E's decision node following entry (the initial node of the subgame following entry), the beliefs $\mu_{\bar{\sigma}}$ associated with $\bar{\sigma}$ at information set H_I are equal to

$$\mu_{\bar{\sigma}}(x) = \frac{\text{Prob}\,(x \mid \bar{\sigma})}{\text{Prob}\,(H_I \mid \bar{\sigma})} = \frac{\text{Prob}\,(x \mid z, \bar{\sigma})\ \text{Prob}\,(z \mid \bar{\sigma})}{\text{Prob}\,(H_I \mid z, \bar{\sigma})\ \text{Prob}\,(z \mid \bar{\sigma})},$$

where $\text{Prob}\,(x \mid z, \bar{\sigma})$ is the probability of reaching node x under strategies $\bar{\sigma}$ conditional on having reached node z. Canceling terms and noting that $\text{Prob}\,(H_I \mid z, \bar{\sigma}) = 1$, we then have $\mu_{\bar{\sigma}}(x) = \text{Prob}\,(x \mid z, \bar{\sigma})$. But this is exactly the probability that firm E plays the action that leads to node x in strategy $\bar{\sigma}$. Thus, any sequence of totally mixed strategies $\{\bar{\sigma}^k\}_{k=1}^{\infty}$ that converge to σ must generate limiting beliefs for firm I that coincide with the play at node z specified in firm E's actual strategy σ_E. It is then immediate that the strategies in any sequential equilibrium must specify Nash equilibrium behavior in this post-entry subgame and thus must constitute a subgame perfect Nash equilbrium. ■

Proposition 9.C.2 gives a general result on the relation between sequential equilibria and subgame perfect Nash equilibria.

Proposition 9.C.2: In every sequential equilibrium (σ, μ) of an extensive form game Γ_E, the equilibrium strategy profile σ constitutes a subgame perfect Nash equilibrium of Γ_E.

Thus, the sequential equilibrium concept strengthens both the SPNE and the weak PBE concepts; every sequential equilibrium is both a weak PBE and an SPNE.

Although the concept of sequential equilibrium restricts beliefs that are off the equilibrium path enough to take care of the problems with the weak PBE concept illustrated in Examples 9.C.4 and 9.C.5, there are some ways in which the requirements on off-equilibrium-path beliefs embodied in the notion of sequential equilibrium may be too strong. For example, they imply that any two players with the same information must have exactly the same beliefs regarding the deviations by other players that have caused play to reach a given part of the game tree.

In Appendix B, we briefly describe another related (and still stronger) solution concept, an *extensive form trembling-hand perfect Nash equilibrium*, first proposed by Selten (1975).[14]

9.D Reasonable Beliefs and Forward Induction

In Section 9.C, we saw the importance of beliefs at unreached information sets for testing the sequential rationality of a strategy. Although the weak perfect Bayesian equilibrium concept and the related stronger concepts discussed in Section 9.C can help rule out noncredible threats, in many games we can nonetheless justify a large range of off-equilibrium-path behavior by picking off-equilibrium-path beliefs appropriately (we shall see some examples shortly). This has led to a considerable amount of recent research aimed at specifying additional restrictions that "reasonable" beliefs should satisfy. In this section, we provide a brief introduction to these ideas. (We shall encounter them again when we study signaling models in Chapter 13, particularly in Appendix A of that chapter.)

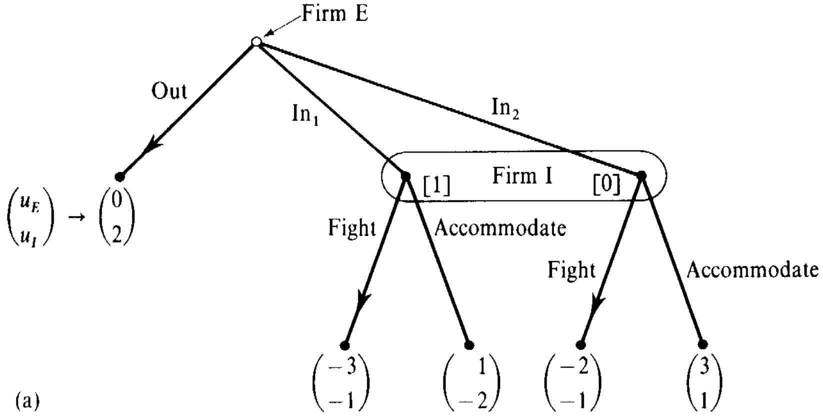
To start, consider the two games depicted in Figure 9.D.1. The first is a variant of the entry game of Figure 9.C.1 in which firm I would now find it worthwhile to fight if it knew that the entrant chose strategy "in_1"; the second is a variant of the Niche Choice game of Example 9.B.4, in which firm E now targets a niche at the time of its entry. Also shown in each diagram is a weak perfect Bayesian equilibrium (arrows denote pure strategy choices, and the numbers in brackets in firm I's information set denote beliefs).

One can argue that in neither game is the equilibrium depicted very sensible.[15] Consider the game in Figure 9.D.1(a). In the weak PBE depicted, if entry occurs, firm I plays "fight" because it believes that firm E has chosen "in_1." But "in_1" is strictly dominated for firm E by "in_2." Hence, it seems reasonable to think that if firm E decided to enter, it must have used strategy "in_2." Indeed, as is commonly done in this literature, one can imagine firm E making the following speech upon entering: "I have entered, but notice that I would never have used 'in_1' to do so because 'in_2' is always a better entry strategy for me. Think about this carefully before you choose your strategy."

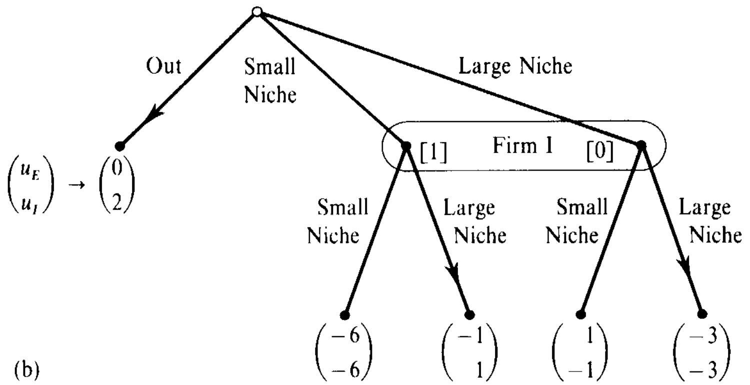
A similar argument holds for the weak PBE depicted in Figure 9.D.1(b). Here "small niche" is strictly dominated for firm E, not by "large niche", but by "out." Once again, firm I could not reasonably hold the beliefs that are depicted. In this case, firm I should recognize that if firm E entered rather than playing "out," it must have chosen the large niche. Now you can imagine firm E saying: "Notice that the only way I could ever do better by entering than by choosing 'out' is by targeting the large niche."

14. Selten actually gave it the name *trembling-hand perfect Nash equilibrium*; we add the modifier *extensive form* to help distinguish it from the normal form concept introduced in Section 8.F.

15. For simplicity, we focus on weak perfect Bayesian equilibria here. The points to be made apply as well to the stronger related notions discussed in Section 9.C. In fact, all the weak perfect Bayesian equilibria discussed here are also sequential equilibria; indeed, they are even extensive form trembling-hand perfect.



(a)



(b)

Figure 9.D.1
Two weak PBEs with unreasonable beliefs.

These arguments make use of what is known as *forward induction* reasoning [see Kohlberg (1989) and Kohlberg and Mertens (1986)]. In using backward induction, a player decides what is an optimal action for her at some point in the game tree based on her calculations of the actions that her opponents will rationally play at *later* points of the game. In contrast, in using forward induction, a player reasons about what could have rationally happened *previously*. For example, here firm I decides on its optimal post-entry action by assuming that firm E must have behaved rationally in its entry decision.

This type of idea is sometimes extended to include arguments based on *equilibrium domination*. For example, suppose that we augment the game in Figure 9.D.1(b) by also giving firm I a move after firm E plays "out," as depicted in Figure 9.D.2 (perhaps "out" really involves entry into some alternative market of firm I's in which firm E has only one potential entry strategy).

The figure depicts a weak PBE of this game in which firm E plays "out" and firm I believes that firm E has chosen "small niche" whenever its post-entry information set is reached. In this game, "small niche" is no longer strictly dominated for firm E by "out," so our previous argument does not apply. Nevertheless, if firm E deviates from this equilibrium by entering, we can imagine firm I thinking that since firm E could have received a payoff of 0 by following its equilibrium strategy, it must be hoping to do better than that by entering, and so it must

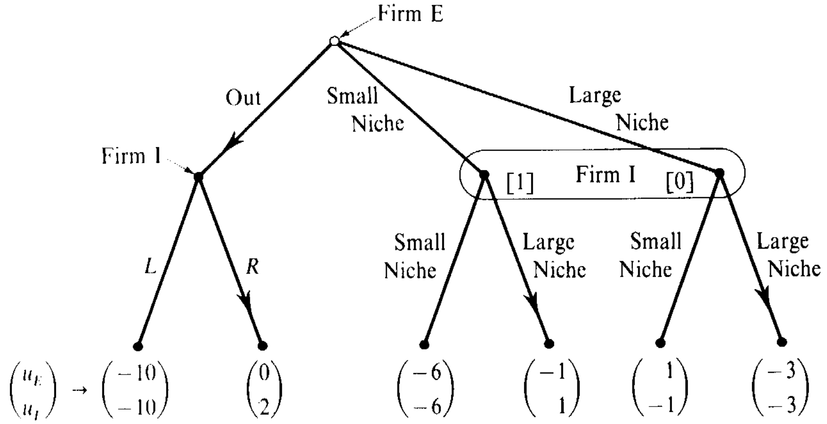


Figure 9.D.2
Strategy "small niche" is equilibrium dominated for firm E.

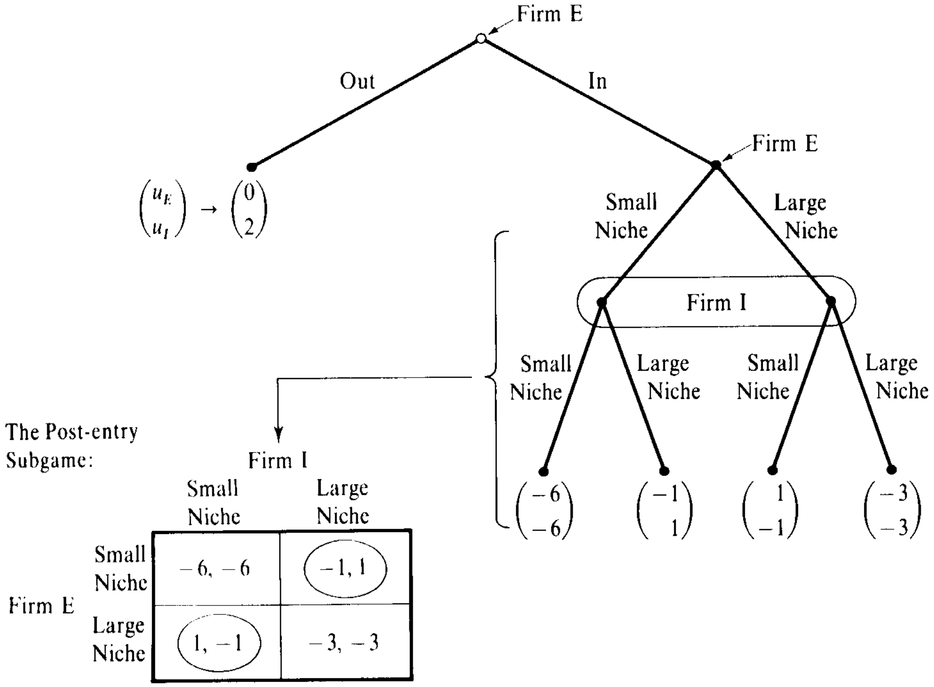


Figure 9.D.3
Forward induction selects equilibrium (large niche, small niche) in the post-entry subgame.

have chosen to target the large niche. In this case, we say that "small niche" is *equilibrium dominated* for firm E; that is, it is dominated if firm E treats its equilibrium payoff as something that it can achieve with certainty by following its equilibrium strategy. (This type of argument is embodied in the *intuitive criterion* refinement that we discuss in Section 13.C and Appendix A of Chapter 13 in the context of signaling models.)

Forward induction can be quite powerful. For example, reconsider the original Niche Choice game depicted in Figure 9.D.3. Recall that there are two (pure strategy) Nash equilibria in the post-entry subgame: (large niche, small niche) and (small niche, large niche). However, the force of the forward induction argument for the game in Figure 9.D.1(b) seems to apply equally well here: Strategy (in, small niche if in) is strictly dominated for firm E by playing "out." As a result, the incumbent should reason that if firm E has played "in," it intends to target the large niche in the post-entry game. If so, firm I is better off targeting the small niche. Thus, forward induction rules out one of the two Nash equilibria in the post-entry subgame.

Although these arguments may seem very appealing, there are also some potential problems. For example, suppose that we are in a world where players make mistakes with some small probability. In such a world, are the forward induction arguments just given convincing? Perhaps not. To see why, suppose that firm E enters in the game shown in Figure 9.D.1(a) when it was supposed to play "out." Now firm I can explain the deviation to itself as being the result of a mistake on firm E's part, a mistake that might equally well have led firm E to pick "in_1" as "in_2." And firm E's speech may not fall on very sympathetic ears: "Of course, firm E is telling me this," reasons the incumbent, "it has made a mistake and now is trying to make the best of it by convincing me to accommodate."

To see this in an even more striking manner, consider the game in Figure 9.D.3. Now, after firm E has entered and the two firms are about to play the simultaneous-move post-entry game, firm E makes its speech. But the incumbent retorts: "Forget it! I think you just made a mistake and even if you did not, I'm going to target the large niche!"

Clearly, the issues here, although interesting and important, are also tricky.

A noticeable feature of these forward induction arguments is how they use the normal form notion of dominance to restrict predicted play in dynamic games. This stands in sharp contrast with our discussion earlier in this chapter, which relied exclusively on the extensive form to determine how players should play in dynamic games. This raises a natural question: Can we somehow use the normal form representation to predict play in dynamic games?

There are at least two reasons why we might think we can. First, as we discussed in Chapter 7, it seems appealing as a matter of logic to think that players simultaneously choosing their strategies in the normal form (e.g., submitting contingent plans to a referee) is equivalent to their actually playing out the game dynamically as represented in the extensive form. Second, in many circumstances, it seems that the notion of weak dominance can get at the idea of sequential rationality. For example, for finite games of perfect information in which no player has equal payoffs at any two terminal nodes, any strategy profile surviving a process of iterated deletion of weakly dominated strategies leads to the same predicted outcome as the SPNE concept (take a look at Example 9.B.1, and see Exercise 9.D.1).

The argument for using the normal form is also bolstered by the fact that extensive form concepts such as weak PBE can be sensitive to what may seem like irrelevant changes in the extensive form. For example, by breaking up firm E's decision in the game in Figure 9.D.1(a) into an "out" or "in" decision followed by an "in_1" or "in_2" decision [just as we did in Figure 9.D.3 for the game in Figure 9.D.1(b)], the unique SPNE (and, hence, the unique sequential equilibrium) becomes firm E entering and playing "in_2" and firm I accommodating. However, the reduced normal form associated with these two games (i.e., the normal form where we eliminate all but one of a player's strategies that have identical payoffs) is invariant to this change in the extensive form; therefore, any solution based on the (reduced) normal form would be unaffected by this change.

These points have led to a renewed interest in the use of the normal form as a device for predicting play in dynamic games [see, in particular, Kohlberg and Mertens (1986)]. At the same time, this issue remains controversial. Many game theorists believe that there is a loss of some information of strategic importance in going from the extensive form to the more condensed normal form. For example, are the games in Figures 9.D.3 and 9.D.1(b) really the same? If you were firm I, would you be as likely to rely on the forward induction argument in the game in Figure 9.D.3 as in that in Figure 9.D.1(b)? Does it matter for your answer whether in the game in Figure 9.D.3 a minute or a month passes between firm E's two decisions? These issues remain to be sorted out.

APPENDIX A: FINITE AND INFINITE HORIZON BILATERAL BARGAINING

In this appendix we study two models of bilateral bargaining as an economically important example of the use of the subgame perfect Nash equilibrium concept. We begin by studying a finite horizon model of bargaining and then consider its infinite horizon counterpart.

Example 9.AA.1: *Finite Horizon Bilateral Bargaining.* Two players, 1 and 2, bargain to determine the split of v dollars. The rules are as follows: The game begins in period 1; in period 1, player 1 makes an offer of a split (a real number between 0 and v) to player 2, which player 2 may then accept or reject. If she accepts, the proposed split is immediately implemented and the game ends. If she rejects, nothing happens until period 2. In period 2, the players' roles are reversed, with player 2 making an offer to player 1 and player 1 then being able to accept or reject it. Each player has a discount factor of $\delta \in (0, 1)$, so that a dollar received in period t is worth δ^{t-1} in period 1 dollars. However, after some finite number of periods T, if an agreement has not yet been reached, the bargaining is terminated and the players each receive nothing. A portion of the extensive form of this game is depicted in Figure 9.AA.1 [this model is due to Stahl (1972)].

There is a unique subgame perfect Nash equilibrium (SPNE) in this game. To see this, suppose first that T is odd, so that player 1 makes the offer in period T if no previous agreement has been reached. Now, player 2 is willing to accept *any* offer in this period because she will get zero if she refuses and the game is terminated (she is indifferent about accepting an offer of zero). Given this fact, the unique SPNE in the subgame that begins in the final period when no agreement has been previously reached has player 1 offer player 2 zero and player 2 accept.[16] Therefore, the payoffs from equilibrium play in this subgame are $(\delta^{T-1}v, 0)$.

Now consider play in the subgame starting in period $T - 1$ when no previous agreement has been reached. Player 2 makes the offer in this period. In any SPNE, player 1 will accept an offer in period $T - 1$ if and only if it provides her with a payoff of at least $\delta^{T-1}v$, since otherwise she will do better rejecting it and waiting to make an offer in period T (she earns $\delta^{T-1}v$ by doing so). Given this fact, in any SPNE, player 2 must make an offer in period $T - 1$ that gives player 1 a payoff of exactly $\delta^{T-1}v$, and player 1 accepts this offer (note that this is player 2's best offer

16. Note that if player 2 is unwilling to accept an offer of zero, then player 1 has no optimal strategy; she wants to make a strictly positive offer ever closer to zero (since player 1 will accept any strictly positive offer). If the reliance on player 1 accepting an offer over which she is indifferent bothers you, you can convince yourself that the analysis of the game in which offers must be in small increments (pennies) yields exactly the same outcome as that identified in the text as the size of these increments goes to zero.

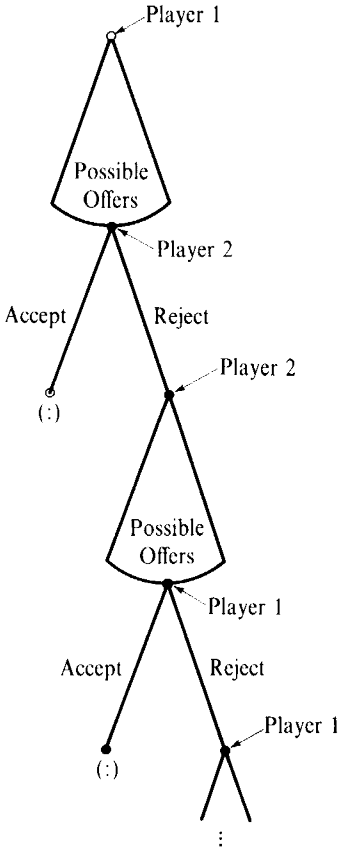


Figure 9.AA.1 The alternating-offer bilateral bargaining game.

among all those that would be accepted, and making an offer that will be rejected is worse for player 2 because it results in her receiving a payoff of zero). The payoffs arising if the game reaches period $T-1$ must therefore be $(\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v)$.

Continuing in this fashion, we can determine that the unique SPNE when T is odd results in an agreement being reached in period 1, a payoff for player 1 of

$$\begin{aligned} v_1^*(T) &= v[1 - \delta + \delta^2 - \cdots + \delta^{T-1}] \\ &= v\left[(1-\delta)\left(\frac{1-\delta^{T-1}}{1-\delta^2}\right) + \delta^{T-1}\right], \end{aligned}$$

and a payoff to player 2 of $v_2^*(T) = v - v_1^*(T)$.

If T is instead even, then player 1 must earn $v - \delta v_1^*(T-1)$ because in any SPNE, player 2 (who will be the first offerer in the odd-number-of-periods subgame that begins in period 2 if she rejects player 1's period 1 offer) will accept an offer in period 1 if and only if it gives her at least $\delta v_1^*(T-1)$, and player 1 will offer her exactly this amount.

Finally, note that as the number of periods grows large ($T \to \infty$), player 1's payoff converges to $v/(1+\delta)$, and player 2's payoff converges to $\delta v/(1+\delta)$. ■

In Example 9.AA.1, the application of the SPNE concept was relatively straightforward; we simply needed to start at the end of the game and work backward. We now consider the infinite horizon counterpart of this game. As we noted in Section

9.B, we can no longer solve for the SPNE in this simple manner when the game has an infinite horizon. Moreover, in many games, introduction of an infinite horizon allows a broad range of behavior to emerge as subgame perfect. Nevertheless, in the infinite horizon bargaining model, the SPNE concept is quite powerful. There is a unique SPNE in this game, and it turns out to be exactly the limiting outcome of the finite horizon model as the length of the horizon T approaches ∞.

Example 9.AA.2: *Infinite Horizon Bilateral Bargaining.* Consider an extension of the finite horizon bargaining game considered in Example 9.AA.1 in which bargaining is no longer terminated after T rounds but, rather, can potentially go on forever. If this happens, the players both earn zero. This model is due to Rubinstein (1982).

We claim that this game has a unique SPNE. In this equilibrium, the players reach an immediate agreement in period 1, with player 1 earning $v/(1+\delta)$ and player 2 earning $\delta v/(1+\delta)$.

The method of analysis we use here, following Shaked and Sutton (1984), makes heavy use of the stationarity of the game (the subgame starting in period 2 looks exactly like that in period 1, but with the players' roles reversed).

To start, let $\bar{v}_1$ denote the largest payoff that player 1 gets in *any* SPNE (i.e., there may, in principle, be multiple SPNEs in this model).[17] Given the stationarity of the model, this is also the largest amount that player 2 can expect in the subgame that begins in period 2 after her rejection of player 1's period 1 offer, a subgame in which player 2 has the role of being the first player to make an offer. As a result, player 1's payoff in any SPNE cannot be lower than the amount $\underline{v}_1 = v - \delta\bar{v}_1$ because, if it was, then player 1 could do better by making a period 1 offer that gives player 2 just slightly more than $\delta\bar{v}_1$. Player 2 is certain to accept any such offer because she will earn only $\delta\bar{v}_1$ by rejecting it (note that we are using subgame perfection here, because we are requiring that the continuation of play after rejection is an SPNE in the continuation subgame and that player 2's response will be optimal given this fact).

Next, we claim that, in any SPNE, $\bar{v}_1$ cannot be larger than $v - \delta\underline{v}_1$. To see this, note that in any SPNE, player 2 is certain to reject any offer in period 1 that gives her less than $\delta\underline{v}_1$ because she can earn at least $\delta\underline{v}_1$ by rejecting it and waiting to make an offer in period 2. Thus, player 1 can do no better than $v - \delta\underline{v}_1$ by making an offer that is accepted in period 1. What about by making an offer that is rejected in period 1? Since player 2 must earn at least $\delta\underline{v}_1$ if this happens, and since agreement cannot occur before period 2, player 1 can earn no more than $\delta v - \delta\underline{v}_1$ by doing this. Hence, we have $\bar{v}_1 \leq v - \delta\underline{v}_1$.

Next, note that these derivations imply that

$$\bar{v}_1 \leq v - \delta\underline{v}_1 = (\underline{v}_1 + \delta\bar{v}_1) - \delta\underline{v}_1,$$

so that

$$\bar{v}_1(1-\delta) \leq \underline{v}_1(1-\delta).$$

Given the definitions of $\underline{v}_1$ and $\bar{v}_1$, this implies that $\underline{v}_1 = \bar{v}_1$, and so player 1's SPNE payoff is uniquely determined. Denote this payoff by v_1°. Since $v_1^\circ = v - \delta v_1^\circ$, we find that player 1 must earn $v_1^\circ = v/(1+\delta)$ and player 2 must earn $v_2^\circ = v - v_1^\circ = \delta v/(1+\delta)$. In addition, recalling the argument in the previous paragraph, we see

17. This maximum can be shown to be well defined, but we will not do so here.

that an agreement will be reached in the first period (player 1 will find it worthwhile to make an offer that player 2 accepts). The SPNE strategies are as follows: A player who has just received an offer accepts it if and only if she is offered at least δv_1°, while a player whose turn it is to make an offer offers exactly δv_1° to the player receiving the offer.

Note that the equilibrium strategies, outcome, and payoffs are precisely the limit of those in the finite game in Example 9.AA.1 as $T \to \infty$. ■

The coincidence of the infinite horizon equilibrium with the limit of the finite horizon equilibria in this model is not a general property of infinite horizon games. The discussion of infinitely repeated games in Chapter 12 provides an illustration of this point.

We should also point out that the outcomes of game-theoretic models of bargaining can be quite sensitive to the precise specification of the bargaining process and players' preferences. Exercises 9.B.7 and 9.B.13 provide an illustration.

APPENDIX B: EXTENSIVE FORM TREMBLING-HAND PERFECT NASH EQUILIBRIUM

In this appendix we extend the analysis presented in Section 9.C by discussing another equilibrium notion that strengthens the consistency conditions on beliefs in the weak PBE concept: *extensive form trembling-hand perfect Nash equilibrium* [due to Selten (1975)]. In fact, this equilibrium concept is the strongest among those discussed in Section 9.C.

The definition of an extensive form trembling-hand perfect Nash equilibrium parallels that for the normal form (see Section 8.F) but has the trembles applied not to a player's mixed strategies, but rather to the player's choice at each of her information sets. A useful way to view this idea is with what Selten (1975) calls the *agent normal form*. This is the normal form that we would derive if we pretended that the player had a set of agents in charge of moving for her at each of her information sets (a different one for each), each acting independently to try to maximize the player's payoff.

Definition 9.BB.1: Strategy profile σ in extensive form game Γ_E is an *extensive form trembling-hand perfect Nash equilibrium* if and only if it is a normal form trembling-hand perfect Nash equilibrium of the agent normal form derived from Γ_E.

To see why it is desirable to have the trembles occurring at each information set rather than over strategies as in the normal-form concept considered in Section 8.F, consider Figure 9.BB.1, which is taken from van Damme (1983). This game has a unique subgame perfect Nash equilibrium: $(\sigma_1, \sigma_2) = ((NR, L), \ell)$. But you can check that $((NR, L), \ell)$ is not the only normal form trembling-hand perfect Nash equilibrium: so are $((R, L), r)$ and $((R, M), r)$. The reason that these two strategy profiles are normal form trembling-hand perfect is that, in the normal form, the tremble to strategy (NR, M) by player 1 can be larger than that to (NR, L) despite the fact that the latter is a better choice for player 1 at her second decision node.

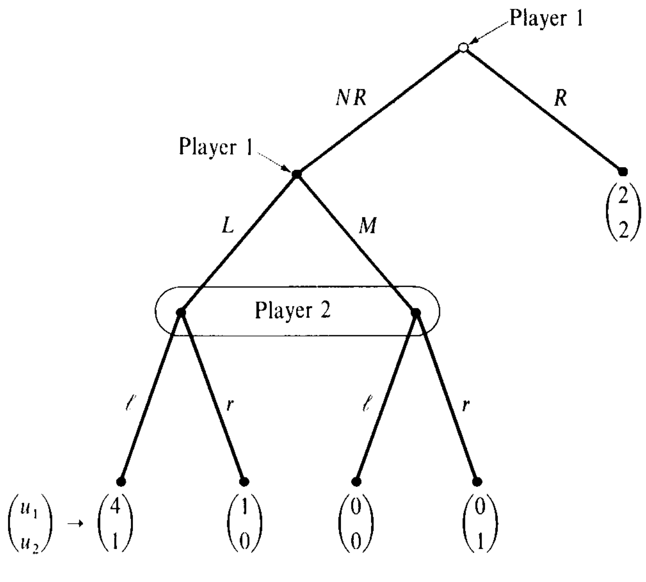


Figure 9.BB.1
Strategy profiles $((R, L), r)$ and $((R, M), r)$ are normal form trembling-hand perfect but are not subgame perfect.

With such a tremble, player 2's best response to player 1's perturbed strategy is r. It is not difficult to see, however, that the unique extensive form trembling-hand perfect Nash equilibrium of this game is $((NR, L), \ell)$ because the agent who moves at player 1's second decision node will put as high a probability as possible on L.

When we compare Definitions 9.BB.1 and 9.C.4, it is apparent that every extensive form trembling-hand perfect Nash equilibrium is a sequential equilibrium. In particular, even though the trembling-hand perfection criterion is not formulated in terms of beliefs, we can use the sequence of (strictly mixed) equilibrium strategies $\{\sigma^k\}_{k=1}^{\infty}$ in the perturbed games of the agent normal form as our strategy sequence for deriving sequential equilibrium beliefs. Because the limiting strategies σ in the extensive form trembling-hand perfect equilibrium are best responses to every element of this sequence, they are also best responses to each other with these derived beliefs. (Every extensive form trembling-hand perfect Nash equilibrium is therefore also subgame perfect.)

In essence, by introducing trembles, the extensive form trembling-hand perfect equilibrium notion makes every part of the tree be reached when strategies are perturbed, and because equilibrium strategies are required to be best responses to perturbed strategies, it insures that equilibrium strategies are sequentially rational. The primary difference between this notion and that of sequential equilibrium is that, like its normal form cousin, the extensive form trembling-hand perfect equilibrium concept can also eliminate some sequential equilibria in which weakly dominated strategies are played. Figure 9.BB.2 (a slight modification of the game in Figure 9.C.1) depicts a sequential equilibrium whose strategies are not extensive form trembling-hand perfect.

In general, however, the concepts are quite close [see Kreps and Wilson (1982) for a formal comparison]; and because it is much easier to check that strategies are best responses at the limiting beliefs than it is to check that they are best responses for a sequence of strategies, sequential equilibrium is much more commonly used. For an interesting further discussion of this concept, consult van Damme (1983).

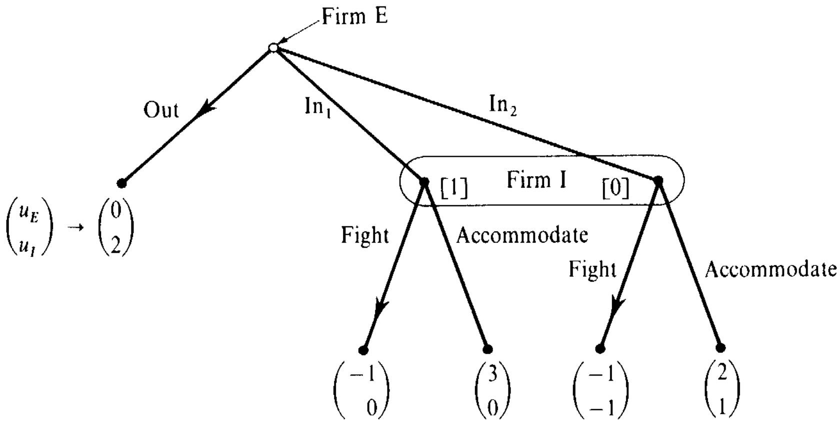


Figure 9.BB.2
A sequential equilibrium need not be extensive form trembling-hand perfect.

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EXERCISES

9.B.1[A] How many subgames are there in the game of Example 9.B.2 (depicted in Figure 9.B.3)?

9.B.2[A] In text.

9.B.3[B] Verify that the strategies identified through backward induction in Example 9.B.2 constitute a Nash equilibrium of the game studied there. Also, identify *all other* pure strategy Nash equilibria of this game. Argue that each of these other equilibria does not satisfy the principle of sequential rationality.

9.B.4[B] Prove that in a finite *zero-sum* game of perfect information, there are unique subgame perfect Nash equilibrium payoffs.

9.B.5[B] (E. Maskin) Consider a game with two players, player 1 and player 2, in which each player i can choose an action from a finite set M_i that contains m_i actions. Player i's payoff if the action choices are (m_1, m_2) is $\phi_i(m_1, m_2)$.

(a) Suppose, first, that the two players move simultaneously. How many strategies does each player have?

(b) Now suppose that player 1 moves first and that player 2 observes player 1's move before choosing her move. How many strategies does each player have?

(c) Suppose that the game in **(b)** has multiple SPNEs. Show that if this is the case, then there exist two pairs of moves (m_1, m_2) and (m'_1, m'_2) (where either $m_1 \neq m'_1$ or $m_2 \neq m'_2$) such that either

$$\text{(i)} \quad \phi_1(m_1, m_2) = \phi_1(m'_1, m'_2)$$

or

$$\text{(ii)} \quad \phi_2(m_1, m_2) = \phi_2(m'_1, m'_2).$$

(d) Suppose that for any two pairs of moves (m_1, m_2) and (m'_1, m'_2) such that $m_1 \neq m'_1$ or $m_2 \neq m'_2$, condition (ii) is violated (i.e., player 2 is never indifferent between pairs of moves). Suppose also that there exists a pure strategy Nash equilibrium in the game in **(a)** in which π_1 is player 1's payoff. Show that in any SPNE of the game in **(b)**, player 1's payoff is at least π_1. Would this conclusion necessarily hold for any *Nash* equilibrium of the game in **(b)**?

(e) Show by example that the conclusion in **(d)** may fail either if condition (ii) holds for some strategy pairs (m_1, m_2), (m'_1, m'_2) with $m_1 \neq m'_1$ or $m_2 \neq m'_2$ or if we replace the phrase *pure strategy Nash equilibrium* with the phrase *mixed strategy Nash equilibrium.*

9.B.6[B] Solve for the mixed strategy equilibrium involving actual randomization in the post-entry subgame of the Niche Choice game in Example 9.B.4. Is there an SPNE that induces this behavior in the post-entry subgame? What are the SPNE strategies?

9.B.7[B] Consider the finite horizon bilateral bargaining game in Appendix A (Example 9.AA.1); but instead of assuming that players discount future payoffs, assume that it costs $c < v$ to make an offer. (Only the player making an offer incurs this cost, and players who have made offers incur this cost even if no agreement is ultimately reached.) What is the (unique) SPNE of this alternative model? What happens as T approaches ∞?

9.B.8[C] Prove that every (finite) game Γ_E has a mixed strategy subgame perfect Nash equilibrium.

9.B.9[B] Consider a game in which the following simultaneous-move game is played twice:

		Player 2 b_1	b_2	b_3
Player 1	a_1	10, 10	2, 12	0, 13
	a_2	12, 2	5, 5	0, 0
	a_3	13, 0	0, 0	1, 1

The players observe the actions chosen in the first play of the game prior to the second play. What are the pure strategy subgame perfect Nash equilibria of this game?

9.B.10[B] Reconsider the game in Example 9.B.3, but now change the post-entry game so that when both players choose "accommodate", instead of receiving the payoffs $(u_E, u_I) = (3, 1)$, the players now must play the following simultaneous-move game:

		Firm I: ℓ	Firm I: r
Firm E	U	3, 1	0, 0
	D	0, 0	x, 3

What are the SPNEs of this game when $x \geq 0$? When $x < 0$?

9.B.11[B] Two firms, A and B, are in a market that is declining in size. The game starts in period 0, and the firms can compete in periods 0, 1, 2, 3, . . . (i.e., indefinitely) if they so choose. Duopoly profits in period t for firm A are equal to $105 - 10t$, and they are $10.5 - t$ for firm B. Monopoly profits (those if a firm is the only one left in the market) are $510 - 25t$ for firm A and $51 - 2t$ for firm B.

Suppose that at the start of each period, each firm must decide either to "stay in" or "exit" if it is still active (they do so simultaneously if both are still active). Once a firm exits, it is out of the market forever and earns zero in each period thereafter. Firms maximize their (undiscounted) sum of profits.

What is this game's subgame perfect Nash equilibrium outcome (and what are the firms' strategies in the equilibrium)?

9.B.12[C] Consider the infinite horizon bilateral bargaining model of Appendix A (Example 9.AA.2). Suppose the discount factors δ_1 and δ_2 of the two players differ. Now what is the (unique) subgame perfect Nash equilibrium?

9.B.13[B] What are the subgame perfect Nash equilibria of the infinite horizon version of Exercise 9.B.7?

9.B.14[B] At time 0, an incumbent firm (firm I) is already in the widget market, and a potential entrant (firm E) is considering entry. In order to enter, firm E must incur a cost of $K > 0$. Firm E's only opportunity to enter is at time 0. There are three production periods. In any period in which both firms are active in the market, the game in Figure 9.Ex.1 is played. Firm E moves first, deciding whether to stay in or exit the market. If it stays in, firm I decides whether to fight (the upper payoff is for firm E). Once firm E plays "out," it is out of

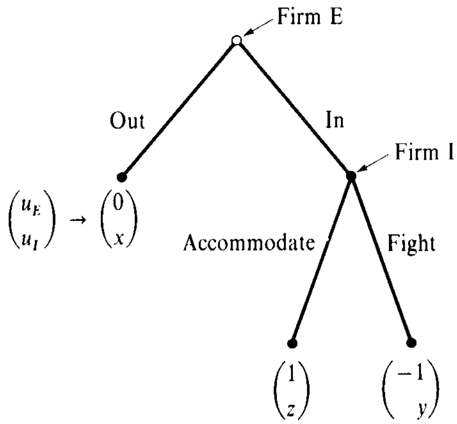

Figure 9.Ex.1

the market forever; firm E earns zero in any period during which it is out of the market, and firm I earns x. The discount factor for both firms is δ.

Assume that:

(A.1) $x > z > y$.
(A.2) $y + \delta x > (1 + \delta)z$.
(A.3) $1 + \delta > K$.

(a) What is the (unique) subgame perfect Nash equilibrium of this game?

(b) Suppose now that firm E faces a financial constraint. In particular, if firm I fights *once* against firm E (in any period), firm E will be forced out of the market from that point on. Now what is the (unique) subgame perfect Nash equilibrium of this game? (If the answer depends on the values of parameters beyond the three assumptions, indicate how.)

9.C.1^B Prove Proposition 9.C.1.

9.C.2^B What is the set of weak PBEs in the game in Example 9.C.3 when $\gamma \in (-1, 0)$?

9.C.3^C A buyer and a seller are bargaining. The seller owns an object for which the buyer has value $v > 0$ (the seller's value is zero). This value is known to the buyer but not to the seller. The value's prior distribution is common knowledge. There are two periods of bargaining. The seller makes a take-it-or-leave-it offer (i.e., names a price) at the start of each period that the buyer may accept or reject. The game ends when an offer is accepted or after two periods, whichever comes first. Both players discount period 2 payoffs with a discount factor of $\delta \in (0, 1)$.

Assume throughout that the buyer always accepts the seller's offer whenever she is indifferent.

(a) Characterize the (pure strategy) weak perfect Bayesian equilibria for a case in which v can take two values v_L and v_H, with $v_H > v_L > 0$, and where $\lambda = \text{Prob}(v_H)$.

(b) Do the same for the case in which v is uniformly distributed on $[\underline{v}, \bar{v}]$.

9.C.4^C A plaintiff, Ms. P, files a suit against Ms. D (the defendant). If Ms. P wins, she will collect π dollars in damages from Ms. D. Ms. D knows the likelihood that Ms. P will win, $\lambda \in [0, 1]$, but Ms. P does not (Ms. D might know if she was actually at fault). They both have strictly positive costs of going to trial of c_p and c_d. The prior distribution of λ has density $f(\lambda)$ (which is common knowledge).

Suppose pretrial settlement negotiations work as follows: Ms. P makes a take-it-or-leave-it settlement offer (a dollar amount) to Ms. D. If Ms. D accepts, she pays Ms. P and the game is over. If she does not accept, they go to trial.

(a) What are the (pure strategy) weak perfect Bayesian equilibria of this game?

(b) What effects do changes in c_p, c_d, and π have?

(c) Now allow Ms. D, after having her offer rejected, to decide not to go to court after all. What are the weak perfect Bayesian equilibria? What about the effects of the changes in **(b)**?

9.C.5^C Reconsider Exercise 9.C.4. Now suppose it is Ms. P who knows λ.

9.C.6^B What are the sequential equilibria in the games in Exercises 9.C.3 to 9.C.5?

9.C.7^B (Based on work by K. Bagwell and developed as an exercise by E. Maskin) Consider the extensive form game depicted in Figure 9.Ex.2.

(a) Find a subgame perfect Nash equilibrium of this game. Is it unique? Are there any other Nash equilibria?

(b) Now suppose that player 2 cannot observe player 1's move. Write down the new extensive form. What is the set of Nash equilibria?

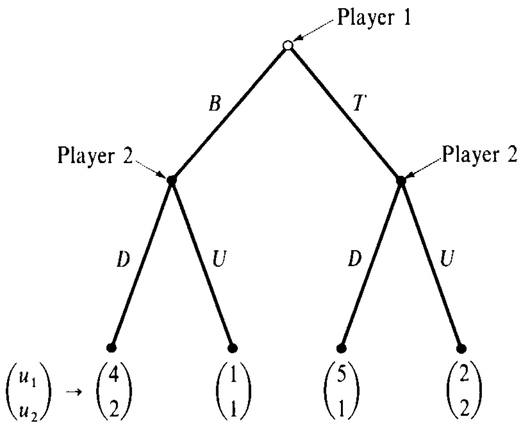

Figure 9.Ex.2

(c) Now suppose that player 2 observes player 1's move correctly with probability $p \in (0, 1)$ and incorrectly with probability $1 - p$ (e.g., if player 1 plays T, player 2 observes T with probability p and observes B with probability $1 - p$). Suppose that player 2's propensity to observe incorrectly (i.e., given by the value of p) is common knowledge to the two players. What is the extensive form now? Show that there is a unique weak perfect Bayesian equilibrium. What is it?

9.D.1[B] Show that under the condition given in Proposition 9.B.2 for existence of a unique subgame perfect Nash equilibrium in a finite game of perfect information, there is an order of iterated removal of weakly dominated strategies for which all surviving strategy profiles lead to the same outcome (i.e., have the same equilibrium path and payoffs) as the subgame perfect Nash equilibrium. [In fact, *any* order of deletion leads to this result; see Moulin (1981).]