Market Equilibrium and Market Failure

In Part III, our focus shifts to the fundamental issue of economics: *the organization of production and the allocation of the resulting commodities among consumers.* This fundamental issue can be addressed from two perspectives, one *positive* and the other *normative*.

From a positive (or *descriptive*) perspective, we can investigate the determination of production and consumption under various institutional mechanisms. The institutional arrangement that is our central focus is that of a *market* (or *private ownership*) *economy*. In a market economy, individual consumers have ownership rights to various assets (such as their labor) and are free to trade these assets in the marketplace for other assets or goods. Likewise, firms, which are themselves owned by consumers, decide on their production plan and trade in the market to secure necessary inputs and sell the resulting outputs. Roughly speaking, we can identify a *market equilibrium* as an outcome of a market economy in which each agent in the economy (i.e., each consumer and firm) is doing as well as he can given the actions of all other agents.

In contrast, from a normative (or *prescriptive*) perspective, we can ask what constitutes a *socially optimal* plan of production and consumption (of course, we will need to be more specific about what "socially optimal" means), and we can then examine the extent to which specific institutions, such as a market economy, perform well in this regard.

In Chapter 10, we study *competitive* (or *perfectly competitive*) *market economies* for the first time. These are market economies in which every relevant good is traded in a market at publicly known prices and all agents act as price takers (recall that much of the analysis of individual behavior in Part I was geared to this case). We begin by defining, in a general way, two key concepts: *competitive* (or *Walrasian*) *equilibrium* and *Pareto optimality* (or *Pareto efficiency*). The concept of competitive equilibrium provides us with an appropriate notion of market equilibrium for competitive market economies. The concept of Pareto optimality offers a minimal and uncontroversial test that any social optimal economic outcome should pass. An economic outcome is said to be Pareto optimal if it is impossible to make some individuals better off without making some other individuals worse off. This concept is a formalization of the idea that there is no waste in society, and it conveniently separates the issue of economic efficiency from more controversial (and political) questions regarding the ideal *distribution* of well-being across individuals.

Chapter 10 then explores these two concepts and the relationships between them in the special context of the *partial equilibrium model.* The partial equilibrium model, which forms the basis for our analysis throughout Part III, offers a considerable analytical simplification; in it, our analysis can be conducted by analyzing a single market (or a small group of related markets) at a time. In this special context, we establish two central results regarding the optimality properties of competitive equilibria, known as the *fundamental theorems of welfare economics.* These can be roughly paraphrased as follows:

> ***The First Fundamental Welfare Theorem.*** If every relevant good is traded in a market at publicly known prices (i.e., if there is a complete set of markets), and if households and firms act perfectly competitively (i.e., as price takers), then the market outcome is Pareto optimal. That is, when markets are complete, *any competitive equilibrium is necessarily Pareto optimal.*

> ***The Second Fundamental Welfare Theorem.*** If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then *any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.*

The first welfare theorem provides a set of conditions under which we can be assured that a market economy will achieve a Pareto optimal result; it is, in a sense, the formal expression of Adam Smith's claim about the "invisible hand" of the market. The second welfare theorem goes even further. It states that under the same set of assumptions as the first welfare theorem plus convexity conditions, *all* Pareto optimal outcomes can in principle be implemented through the market mechanism. That is, a public authority who wishes to implement a particular Pareto optimal outcome (reflecting, say, some political consensus on proper distributional goals) may always do so by appropriately redistributing wealth and then "letting the market work."

In an important sense, the first fundamental welfare theorem establishes the perfectly competitive case as a benchmark for thinking about outcomes in market economies. In particular, any inefficiencies that arise in a market economy, and hence any role for Pareto-improving market intervention, *must* be traceable to a violation of at least one of the assumptions of this theorem.

The remainder of Part III, Chapters 11 to 14, can be viewed as a development of this theme. In these chapters, we study a number of ways in which actual markets may depart from this perfectly competitive ideal and where, as a result, market equilibria fail to be Pareto optimal, a situation known as *market failure.*

In Chapter 11, we study *externalities* and *public goods.* In both cases, the actions of one agent directly affect the utility functions or production sets of other agents in the economy. We see there that the presence of these nonmarketed "goods" or "bads" (which violates the complete markets assumption of the first welfare theorem) undermines the Pareto optimality of market equilibrium.

In Chapter 12, we turn to the study of settings in which some agents in the economy have *market power* and, as a result, fail to act as price takers. Once again, an assumption of the first fundamental welfare theorem fails to hold, and market equilibria fail to be Pareto optimal as a result.

In Chapters 13 and 14, we consider situations in which an *asymmetry of information* exists among market participants. The complete markets assumption of the first welfare theorem implicitly requires that the characteristics of traded commodities be observable by all market participants because, without this observability, distinct markets cannot exist for commodities that have different characteristics. Chapter 13 focuses on the case in which asymmetric information exists between agents at the time of contracting. Our discussion highlights several phenomena—*adverse selection*, *signaling*, and *screening*—that can arise as a result of this informational imperfection, and the welfare loss that it causes. Chapter 14 in contrast, investigates the case of postcontractual asymmetric information, a problem that leads us to the study of the *principal-agent model*. Here, too, the presence of asymmetric information prevents trade of all relevant commodities and can lead market outcomes to be Pareto inefficient.

We rely extensively in some places in Part III on the tools that we developed in Parts I and II. This is particularly true in Chapter 10, where we use material developed in Part I, and Chapters 12 and 13, where we use the game-theoretic tools developed in Part II.

A much more complete and general study of competitive market economies and the fundamental welfare theorems is reserved for Part IV.

CHAPTER 10

Competitive Markets

10.A Introduction

In this chapter, we consider, for the first time, an entire economy in which consumers and firms interact through markets. The chapter has two principal goals: first, to formally introduce and study two key concepts, the notions of *Pareto optimality* and *competitive equilibrium*, and second, to develop a somewhat special but analytically very tractable context for the study of market equilibrium, the *partial equilibrium model*.

We begin in Section 10.B by presenting the notions of a *Pareto optimal* (or *Pareto efficient*) *allocation* and of a *competitive* (or *Walrasian*) *equilibrium* in a general setting.

Starting in Section 10.C, we narrow our focus to the partial equilibrium context. The partial equilibrium approach, which originated in Marshall (1920), envisions the market for a single good (or group of goods) for which each consumer's expenditure constitutes only a small portion of his overall budget. When this is so, it is reasonable to assume that changes in the market for this good will leave the prices of all other commodities approximately unaffected and that there will be, in addition, negligible wealth effects in the market under study. We capture these features in the simplest possible way by considering a two-good model in which the expenditure on all commodities other than that under consideration is treated as a single composite commodity (called the *numeraire* commodity), and in which consumers' utility functions take a quasilinear form with respect to this numeraire. Our study of the competitive equilibria of this simple model lends itself to extensive demand-and-supply graphical analysis. We also discuss how to determine the comparative statics effects that arise from exogenous changes in the market environment. As an illustration, we consider the effects on market equilibrium arising from the introduction of a distortionary commodity tax.

In Section 10.D, we analyze the properties of Pareto optimal allocations in the partial equilibrium model. Most significantly, we establish for this special context the validity of the *fundamental theorems of welfare economics*: Competitive equilibrium allocations are necessarily Pareto optimal, and any Pareto optimal allocation can be achieved as a competitive equilibrium if appropriate lump-sum transfers are made.

As we noted in the introduction to Part III, these results identify an important benchmark case in which market equilibria yield desirable economic outcomes. At the same time, they provide a framework for identifying situations of market failure, such as those we study in Chapters 11 to 14.

In Section 10.E, we consider the measurement of welfare changes in the partial equilibrium context. We show that these can be represented by areas between properly defined demand and supply curves. As an application, we examine the deadweight loss of distortionary taxation.

Section 10.F contemplates settings characterized by *free entry*, that is, settings in which all potential firms have access to the most efficient technology and may enter and exit markets in response to the profit opportunities they present. We define a notion of *long-run competitive equilibrium* and then use it to distinguish between long-run and short-run comparative static effects in response to changes in market conditions.

In Section 10.G, we provide a more extended discussion of the use of partial equilibrium analysis in economic modeling.

The material covered in this chapter traces its roots far back in economic thought. An excellent source for further reading is Stigler (1987). We should emphasize that the analysis of competitive equilibrium and Pareto optimality presented here is very much a first pass. In Part IV we return to the topic for a more complete and general investigation; many additional references will be given there.

10.B Pareto Optimality and Competitive Equilibria

In this section, we introduce and discuss the concepts of *Pareto optimality* (or *Pareto efficiency*) and *competitive* (or *Walrasian*) *equilibrium* in a general setting.

Consider an economy consisting of I consumers (indexed by $i = 1, \dots, I$), J firms (indexed by $j = 1, \dots, J$), and L goods (indexed by $\ell = 1, \dots, L$). Consumer i's preferences over consumption bundles $x_i = (x_{1i}, \dots, x_{Li})$ in his consumption set $X_i \subset \mathbb{R}^L$ are represented by the utility function $u_i(\cdot)$. The total amount of each good $\ell = 1, \dots, L$ initially available in the economy, called the total *endowment* of good ℓ, is denoted by $\omega_\ell \geq 0$ for $\ell = 1, \dots, L$. It is also possible, using the production technologies of the firms, to transform some of the initial endowment of a good into additional amounts of other goods. Each firm j has available to it the production possibilities summarized by the production set $Y_j \subset \mathbb{R}^L$. An element of Y_j is a production vector $y_j = (y_{1j}, \dots, y_{Lj}) \in \mathbb{R}^L$. Thus, if $(y_1, \dots, y_J) \in \mathbb{R}^{LJ}$ are the production vectors of the J firms, the total (net) amount of good ℓ available to the economy is $\omega_\ell + \sum_j y_{\ell j}$ (recall that negative entries in a production vector denote input usage; see Section 5.B).

We begin with Definition 10.B.1, which identifies the set of possible outcomes in this economy:

Definition 10.B.1: An *economic allocation* $(x_1, \dots, x_I, y_1, \dots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \dots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \dots, J$. The allocation $(x_1, \dots, x_I, y_1, \dots, y_J)$ is *feasible* if

$$\sum_{i=1}^{I} x_{\ell i} \leq \omega_\ell + \sum_{j=1}^{J} y_{\ell j} \quad \text{for } \ell = 1, \dots, L.$$

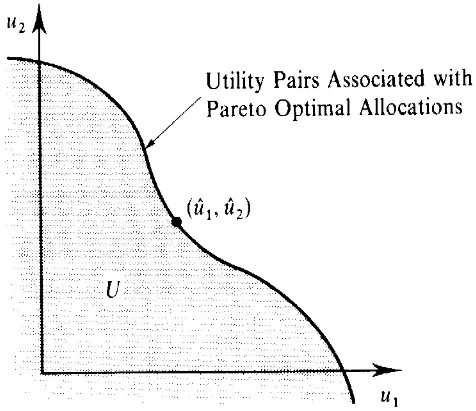


Figure 10.B.1
A utility possibility set.

Thus, an economic allocation is feasible if the total amount of each good consumed does not exceed the total amount available from both the initial endowment and production.

Pareto Optimality

It is often of interest to ask whether an economic system is producing an "optimal" economic outcome. An essential requirement for any optimal economic allocation is that it possess the property of *Pareto optimality* (or *Pareto efficiency*).

Definition 10.B.2: A feasible allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *Pareto optimal* (or *Pareto efficient*) if there is no other feasible allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ such that $u_i(x'_i) \geq u_i(x_i)$ for all $i = 1, \ldots, I$ and $u_i(x'_i) > u_i(x_i)$ for some i.

An allocation that is Pareto optimal uses society's initial resources and technological possibilities efficiently in the sense that there is no alternative way to organize the production and distribution of goods that makes some consumer better off without making some other consumer worse off.

Figure 10.B.1 illustrates the concept of Pareto optimality. There we depict the set of attainable utility levels in a two-consumer economy. This set is known as a *utility possibility set* and is defined in this two-consumer case by

$$U = \{(u_1, u_2) \in \mathbb{R}^2\text{: there exists a feasible allocation } (x_1, x_2, y_1, \ldots, y_J) \text{ such that } u_i \leq u_i(x_i) \text{ for } i = 1, 2\}.$$

The set of Pareto optimal allocations corresponds to those allocations that generate utility pairs lying in the utility possibility set's northeast boundary, such as point $(\hat{u}_1, \hat{u}_2)$. At any such point, it is impossible to make one consumer better off without making the other worse off.

It is important to note that the criterion of Pareto optimality does not insure that an allocation is in any sense equitable. For example, using all of society's resources and technological capabilities to make a single consumer as well off as possible, subject to all other consumers receiving a subsistence level of utility, results in an allocation that is Pareto optimal but not in one that is very desirable on distributional grounds. Nevertheless, Pareto optimality serves as an important minimal test for the desirability of an allocation; it does, at the very least, say that there is no waste in the allocation of resources in society.

Competitive Equilibria

Throughout this chapter, we are concerned with the analysis of competitive market economies. In such an economy, society's initial endowments and technological possibilities (i.e., the firms) are owned by consumers. We suppose that consumer i initially owns $\omega_{\ell i}$ of good ℓ, where $\sum_i \omega_{\ell i} = \omega_\ell$. We denote consumer i's vector of endowments by $\omega_i = (\omega_{1i}, \ldots, \omega_{Li})$. In addition, we suppose that consumer i owns a share θ_{ij} of firm j (where $\sum_i \theta_{ij} = 1$), giving him a claim to fraction θ_{ij} of firm j's profits.

In a competitive economy, a market exists for each of the L goods, and all consumers and producers act as price takers. The idea behind the price-taking assumption is that if consumers and producers are small relative to the size of the market, they will regard market prices as unaffected by their own actions.[1]

Denote the vector of market prices for goods $1, \ldots, L$ by $p = (p_1, \ldots, p_L)$. Definition 10.B.3 introduces the notion of a competitive (or Walrasian) equilibrium.

Definition 10.B.3: The allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and price vector $p^* \in \mathbb{R}^L$ constitute a *competitive* (or *Walrasian*) *equilibrium* if the following conditions are satisfied:

(i) *Profit maximization*: For each firm j, y_j^* solves

$$\underset{y_j \in Y_j}{\text{Max}} \quad p^* \cdot y_j. \tag{10.B.1}$$

(ii) *Utility maximization*: For each consumer i, x_i^* solves

$$\underset{x_i \in X_i}{\text{Max}} \quad u_i(x_i) \tag{10.B.2}$$

$$\text{s.t. } p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot y_j^*).$$

(iii) *Market clearing*: For each good $\ell = 1, \ldots, L$,

$$\sum_{i=1}^{I} x_{\ell i}^* = \omega_\ell + \sum_{j=1}^{J} y_{\ell j}^*. \tag{10.B.3}$$

Definition 10.B.3 delineates three sorts of conditions that must be met for a competitive economy to be considered to be in equilibrium. Conditions (i) and (ii) reflect the underlying assumption, common to nearly all economic models, that agents in the economy seek to do as well as they can for themselves. Condition (i) states that each firm must choose a production plan that maximizes its profits, taking as given the equilibrium vector of prices of its outputs and inputs (for the justification of the profit-maximization assumption, see Section 5.G). We studied this competitive behavior of the firm extensively in Chapter 5.

Condition (ii) requires that each consumer chooses a consumption bundle that maximizes his utility given the budget constraint imposed by the equilibrium prices and by his wealth. We studied this competitive behavior of the consumer extensively in Chapter 3. One difference here, however, is that the consumer's wealth is now a function of prices. This dependence of wealth on prices arises in

1. Strictly speaking, it is *equilibrium* market prices that they will regard as unaffected by their actions. For more on this point, see the small-type discussion later in this section.

two ways: First, prices determine the value of the consumer's initial endowments; for example, an individual who initially owns real estate is poorer if the price of real estate falls. Second, the equilibrium prices affect firms' profits and hence the value of the consumer's shareholdings.

Condition (iii) is somewhat different. It requires that, at the equilibrium prices, the desired consumption and production levels identified in conditions (i) and (ii) are in fact mutually compatible; that is, the aggregate supply of each commodity (its total endowment plus its net production) equals the aggregate demand for it. If excess supply or demand existed for a good at the going prices, the economy could not be at a point of equilibrium. For example, if there is excess demand for a particular commodity at the existing prices, some consumer who is not receiving as much of the commodity as he desires could do better by offering to pay just slightly more than the going market price and thereby get sellers to offer the commodity to him first. Similarly, if there is excess supply, some seller will find it worthwhile to offer his product at a slight discount from the going market price.[2]

Note that in justifying why an equilibrium must involve no excess demand or supply, we have actually made use of the fact that consumers and producers *might not* simply take market prices as given. How are we to reconcile this argument with the underlying price-taking assumption?

An answer to this apparent paradox comes from recognizing that consumers and producers *always* have the ability to alter their offered prices (in the absence of any institutional constraints preventing this). For the price-taking assumption to be appropriate, what we want is that they have no *incentive* to alter prices that, if taken as given, equate demand and supply (we have already seen that they *do* have an incentive to alter prices that do not equate demand and supply).

Notice that as long as consumers can make their desired trades at the going market prices, they will not wish to offer more than the market price to entice sellers to sell to them first. Similarly, if producers are able to make their desired sales, they will have no incentive to undercut the market price. Thus, at a price that equates demand and supply, consumers do not wish to raise prices, and firms do not wish to lower them.

More troublesome is the possibility that a buyer might try to lower the price he pays or that a seller might try to raise the price he charges. A seller, for example, may possess the ability to raise profitably prices of the goods he sells above their competitive level (see Chapter 12). In this case, there is no reason to believe that this market power will not be exercised. To rescue the price-taking assumption, one needs to argue that under appropriate (competitive) conditions such market power does not exist. This we do in Sections 12.F and 18.C, where we formalize the idea that if market participants' desired trades are small relative to the size of the market, then they will have little incentive to depart from market prices. Thus, in a suitably defined equilibrium, they will act approximately like price takers.

Note from Definition 10.B.3 that if the allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and price vector $p^* \gg 0$ constitute a competitive equilibrium, then so do the allocation

2. Strictly speaking, this second part of the argument requires the price to be positive; indeed, if the price is zero (i.e., if the good is free), then excess supply should be permissible at equilibrium. In the remainder of this chapter, however, consumer preferences will be such as to preclude this possibility (goods will be assumed to be desirable). Hence, we neglect this possibility here.

$(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and price vector $\alpha p^* = (\alpha p_1^*, \ldots, \alpha p_L^*)$ for any scalar $\alpha > 0$ (see Exercise 10.B.2). As a result, we can normalize prices without loss of generality. In this chapter, we always normalize by setting one good's price equal to 1.

Lemma 10.B.1 will also prove useful in identifying competitive equilibria.

Lemma 10.B.1: If the allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ and price vector $p \gg 0$ satisfy the market clearing condition (10.B.3) for all goods $\ell \neq k$, and if every consumer's budget constraint is satisfied with equality, so that $p \cdot x_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j$ for all i, then the market for good k also clears.

Proof: Adding up the consumers' budget constraints over the I consumers and rearranging terms, we get

$$\sum_{\ell \neq k} p_\ell \left(\sum_{i=1}^{I} x_{\ell i} - \omega_\ell - \sum_{j=1}^{J} y_{\ell j} \right) = -p_k \left(\sum_{i=1}^{I} x_{ki} - \omega_k - \sum_{j=1}^{J} y_{kj} \right).$$

By market clearing in goods $\ell \neq k$, the left-hand side of this equation is equal to zero. Thus, the right-hand side must be equal to zero as well. Because $p_k > 0$, this implies that we have market clearing in good k. ■

In the models studied in this chapter, Lemma 10.B.1 will allow us to identify competitive equilibria by checking for market clearing in only $L - 1$ markets. Lemma 10.B.1 is really just a matter of double-entry accountancy. If consumers' budget constraints hold with equality, the dollar value of each consumer's planned purchases equals the dollar value of what he plans to sell plus the dollar value of his share (θ_{ij}) of the firms' (net) supply, and so the total value of planned purchases in the economy must equal the total value of planned sales. If those values are equal to each other in all markets but one, then equality must hold in the remaining market as well.

10.C Partial Equilibrium Competitive Analysis

Marshallian partial equilibrium analysis envisions the market for one good (or several goods, as discussed in Section 10.G) that constitutes a small part of the overall economy. The small size of the market facilitates two important simplifications for the analysis of market equilibrium:[3] First, as Marshall (1920) emphasized, when the expenditure on the good under study is a small portion of a consumer's total expenditure, only a small fraction of any additional dollar of wealth will be spent on this good; consequently, we can expect wealth effects for it to be small. Second, with similarly dispersed substitution effects, the small size of the market under study should lead the prices of other goods to be approximately unaffected by changes in this market.[4] Because of this fixity of other prices, we are justified in treating the expenditure on these other goods as a single composite commodity, which we call the *numeraire* (see Exercise 3.G.5).

3. The following points have been formalized by Vives (1987). (See Exercise 10.C.1 for an illustration.)

4. This is not the only possible justification for taking other goods' prices as being unaffected by the market under study; see Section 10.G.

With this partial equilibrium interpretation as our motivation, we proceed to study a simple two-good quasilinear model. There are two commodities: good ℓ and the numeraire. We let x_i and m_i denote consumer i's consumption of good ℓ and the numeraire, respectively. Each consumer $i = 1, \ldots, I$ has a utility function that takes the quasilinear form (see Sections 3.B and 3.C):

$$u_i(m_i, x_i) = m_i + \phi_i(x_i).$$

We let each consumer's consumption set be $\mathbb{R} \times \mathbb{R}_+$, and so we assume for convenience that consumption of the numeraire commodity m can take negative values. This is to avoid dealing with boundary problems. We assume that $\phi_i(\cdot)$ is bounded above and twice differentiable, with $\phi_i'(x_i) > 0$ and $\phi_i''(x_i) < 0$ at all $x_i \geq 0$. We normalize $\phi_i(0) = 0$.

In terms of our partial equilibrium interpretation, we think of good ℓ as the good whose market is under study and of the numeraire as representing the composite of all other goods (m stands for the total money expenditure on these other goods). Recall that with quasilinear utility functions, wealth effects for non-numeraire commodities are null.

In the discussion that follows, we normalize the price of the numeraire to equal 1, and we let p denote the price of good ℓ.

Each firm $j = 1, \ldots, J$ in this two-good economy is able to produce good ℓ from good m. The amount of the numeraire required by firm j to produce $q_j \geq 0$ units of good ℓ is given by the cost function $c_j(q_j)$ (recall that the price of the numeraire is 1). Letting z_j denote firm j's use of good m as an input, its production set is therefore

$$Y_j = \{(-z_j, q_j)\colon\ q_j \geq 0 \text{ and } z_j \geq c_j(q_j)\}.$$

In what follows, we assume that $c_j(\cdot)$ is twice differentiable, with $c_j'(q_j) > 0$ and $c_j''(q_j) \geq 0$ at all $q_j \geq 0$. [In terms of our partial equilibrium interpretation, we can think of $c_j(q_j)$ as actually arising from some multiple-input cost function $c_j(\bar{w}, q_j)$, given the fixed vector of factor prices $\bar{w}$.[5]]

For simplicity, we shall assume that there is no initial endowment of good ℓ, so that all amounts consumed must be produced by the firms. Consumer i's initial endowment of the numeraire is the scalar $\omega_{mi} > 0$, and we let $\omega_m = \sum_i \omega_{mi}$.

We now proceed to identify the competitive equilibria for this two-good quasilinear model. Applying Definition 10.B.3, we consider first the implications of profit and utility maximization.

Given the price p^* for good ℓ, firm j's equilibrium output level q_j^* must solve

$$\underset{q_j \geq 0}{\text{Max}} \quad p^* q_j - c_j(q_j),$$

which has the necessary and sufficient first-order condition

$$p^* \leq c_j'(q_j^*), \quad \text{with equality if } q_j^* > 0.$$

On the other hand, consumer i's equilibrium consumption vector (m_i^*, x_i^*) must

5. Some of the exercises at the end of the chapter investigate the effects of exogenous changes in these factor prices.

solve

$$\begin{aligned} \underset{m_i \in \mathbb{R},\, x_i \in \mathbb{R}_+}{\text{Max}} \quad & m_i + \phi_i(x_i) \\ \text{s.t.} \quad & m_i + p^* x_i \le \omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)). \end{aligned}$$

In any solution to this problem, the budget constraint holds with equality. Substituting for m_i from this constraint, we can rewrite consumer i's problem solely in terms of choosing his optimal consumption of good ℓ. Doing so, we see that x_i^* must solve

$$\underset{x_i \ge 0}{\text{Max}} \quad \phi_i(x_i) - p^* x_i + \left[\omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) \right],$$

which has the necessary and sufficient first-order condition

$$\phi_i'(x_i^*) \le p^*, \quad \text{with equality if } x_i^* > 0.$$

In what follows, it will be convenient to adopt the convention of identifying an equilibrium allocation by the levels of good ℓ consumed and produced, $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$, with the understanding that consumer i's equilibrium consumption of the numeraire is then $m_i^* = [\omega_{mi} + \sum_j \theta_{ij}(p^* q_j^* - c_j(q_j^*))] - p^* x_i^*$ and that firm j's equilibrium usage of the numeraire as an input is $z_j^* = c_j(q_j^*)$.

To complete the development of the equilibrium conditions for this model, recall that by Lemma 10.B.1, we need only check that the market for good ℓ clears.[6] Hence, we conclude that the allocation $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and the price p^* constitute a competitive equilibrium if and only if

$$p^* \le c_j'(q_j^*), \quad \text{with equality if } q_j^* > 0 \qquad j = 1, \ldots, J. \tag{10.C.1}$$

$$\phi_i'(x_i^*) \le p^*, \quad \text{with equality if } x_i^* > 0 \qquad i = 1, \ldots, I. \tag{10.C.2}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*. \tag{10.C.3}$$

At any interior solution, condition (10.C.1) says that firm j's marginal benefit from selling an additional unit of good ℓ, p^*, exactly equals its marginal cost $c_j'(q_j^*)$. Condition (10.C.2) says that consumer i's marginal benefit from consuming an additional unit of good ℓ, $\phi_i'(x_i^*)$, exactly equals its marginal cost p^*. Condition (10.C.3) is the market-clearing equation. Together, these $I + J + 1$ conditions characterize the $(I + J + 1)$ equilibrium values $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and p^*. Note that as long as $\text{Max}_i\, \phi_i'(0) > \text{Min}_j\, c_j'(0)$, the aggregate consumption and production of good ℓ must be strictly positive in a competitive equilibrium [this follows from conditions (10.C.1) and (10.C.2)]. For simplicity, we assume that this is the case in the discussion that follows.

Conditions (10.C.1) to (10.C.3) have a very important property: They do not involve, in any manner, the endowments or the ownership shares of the consumers. As a result, we see that *the equilibrium allocation and price are independent of the*

6. Note that we must have $p^* > 0$ in any competitive equilibrium; otherwise, consumers would demand an infinite amount of good ℓ [recall that $\phi_i'(\cdot) > 0$].

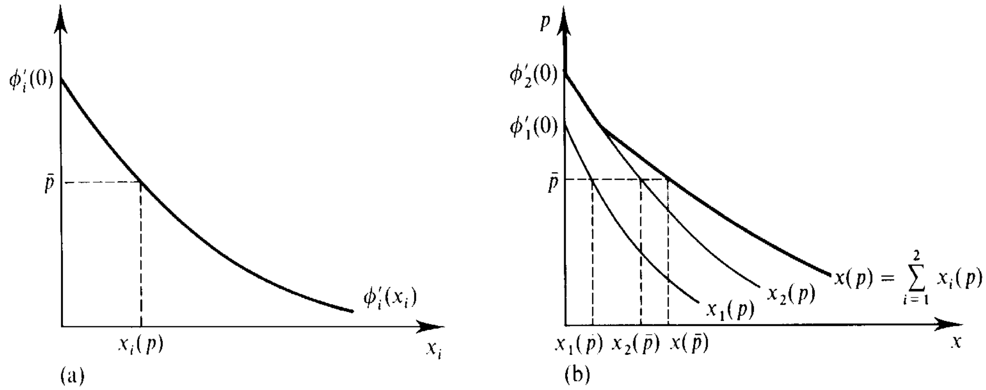


Figure 10.C.1
Construction of the aggregate demand function.
(a) Determination of consumer i's demand.
(b) Construction of the aggregate demand function ($I = 2$).

distribution of endowments and ownership shares. This important simplification arises from the quasilinear form of consumer preferences.[7]

The competitive equilibrium of this model can be nicely represented using the traditional Marshallian graphical technique that identifies the equilibrium price as the point of intersection of aggregate demand and aggregate supply curves.

We can derive the aggregate demand function for good ℓ from condition (10.C.2). Because $\phi_i''(\cdot) < 0$ and $\phi_i(\cdot)$ is bounded, $\phi_i'(\cdot)$ is a strictly decreasing function of x_i taking all values in the set $(0, \phi_i'(0)]$. Therefore, for each possible level of $p > 0$, we can solve for a unique level of x_i, denoted $x_i(p)$, that satisfies condition (10.C.2). Note that if $p \geq \phi_i'(0)$, then $x_i(p) = 0$. Figure 10.C.1(a) depicts this construction for a price $\bar{p} > 0$. The function $x_i(\cdot)$ is consumer i's *Walrasian demand function* for good ℓ (see Section 3.D) which, because of quasilinearity, does not depend on the consumer's wealth. It is continuous and nonincreasing in p at all $p > 0$, and is strictly decreasing at any $p < \phi_i'(0)$ [at any such p, we have $x_i'(p) = 1/\phi_i''(x_i(p)) < 0$].

The *aggregate demand function for good ℓ* is then the function $x(p) = \sum_i x_i(p)$, which is continuous and nonincreasing at all $p > 0$, and is strictly decreasing at any $p < \text{Max}_i\, \phi_i'(0)$. Its construction is depicted in Figure 10.C.1(b) for the case in which $I = 2$; it is simply the horizontal summation of the individual demand functions and is drawn in the figure with a heavy trace. Note that $x(p) = 0$ whenever $p \geq \text{Max}_i\, \phi_i'(0)$.

The aggregate supply function can be similarly derived from condition (10.C.1).[8] Suppose, first, that every $c_j(\cdot)$ is strictly convex and that $c_j'(q_j) \to \infty$ as $q_j \to \infty$. Then, for any $p > 0$, we can let $q_j(p)$ denote the unique level of q_j that satisfies condition (10.C.1). Note that for $p \leq c_j'(0)$, we have $q_j(p) = 0$. Figure 10.C.2(a) illustrates this construction for a price $\bar{p} > 0$. The function $q_j(\cdot)$ is firm j's *supply function* for good ℓ (see Sections 5.C and 5.D). It is continuous and nondecreasing at all $p > 0$, and is strictly increasing at any $p > c_j'(0)$ [for any such p, $q_j'(p) = 1/c_j''(q_j(p)) > 0$].

The *aggregate* (or *industry*) *supply function* for good ℓ is then the function $q(p) = \sum_j q_j(p)$, which is continuous and nondecreasing at all $p > 0$, and is strictly increasing at any $p > \text{Min}_j\, c_j'(0)$. Its construction is depicted in Figure 10.C.2(b) for

7. See Section 10.G for a further discussion of this general feature of equilibrium in economies with quasilinear utility functions.

8. See Section 5.D for an extensive discussion of individual supply in the one-input, one-output case.

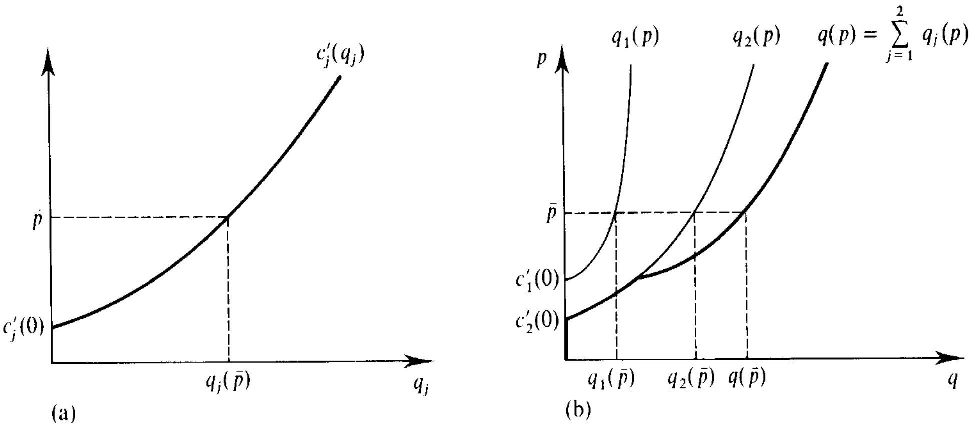


Figure 10.C.2
Construction of the aggregate supply function.
(a) Determination of firm j's supply.
(b) Construction of the aggregate supply function ($J = 2$).

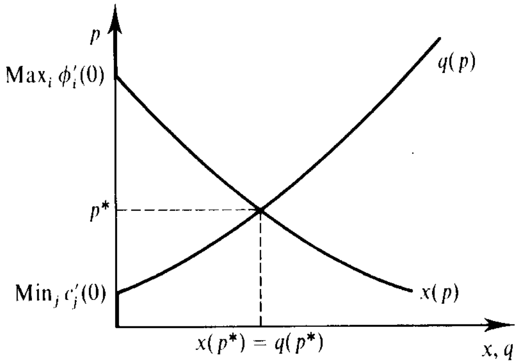


Figure 10.C.3
The equilibrium price equates demand and supply.

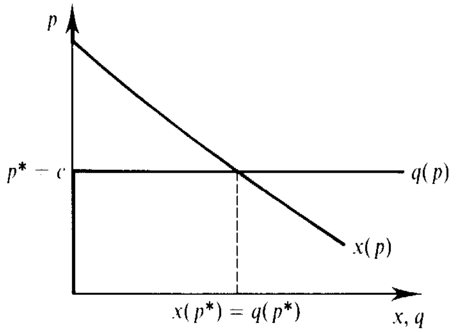
the case in which $J = 2$; it is equal to the horizontal sum of the individual firms' supply functions and is drawn in the figure with a heavy trace. Note that $q(p) = 0$ whenever $p \leq \text{Min}_j\, c_j'(0)$.

To find the equilibrium price of good ℓ, we need only find the price p^* at which aggregate demand equals aggregate supply, that is, at which $x(p^*) = q(p^*)$. When $\text{Max}_i\, \phi_i'(0) > \text{Min}_j\, c_j'(0)$ as we have assumed, at any $p \geq \text{Max}_i\, \phi_i'(0)$ we have $x(p) = 0$ and $q(p) > 0$. Likewise, at any $p \leq \text{Min}_j\, c_j'(0)$ we have $x(p) > 0$ and $q(p) = 0$. The existence of an equilibrium price $p^* \in (\text{Min}_j\, c_j'(0), \text{Max}_i\, \phi_i'(0))$ then follows from the continuity properties of $x(\cdot)$ and $q(\cdot)$. The solution is depicted in Figure 10.C.3. Note also that because $x(\cdot)$ is strictly decreasing at all $p < \text{Max}_i\, \phi_i'(0)$ and $q(\cdot)$ is strictly increasing at all $p > \text{Min}_j\, c_j'(0)$, this equilibrium price is uniquely defined.[9] The individual consumption and production levels of good ℓ in this equilibrium are then given by $x_i^* = x_i(p^*)$ for $i = 1, \ldots, I$ and $q_j^* = q_j(p^*)$ for $j = 1, \ldots, J$.

More generally, if some $c_j(\cdot)$ is merely convex [e.g., if $c_j(\cdot)$ is linear, as in the constant returns case], then $q_j(\cdot)$ is a convex-valued correspondence rather than a function and it may be well defined only on a subset of prices.[10] Nevertheless, the

9. Be warned, however, that the uniqueness of equilibrium is a property that need not hold in more general settings in which wealth effects are present. (See Chapter 17.)

10. For example, if firm j has $c_j(q_j) = c_j q_j$ for some scalar $c_j > 0$, then when $p > c_j$, we have $q_j(p) = \infty$. As a result, if $p > c_j$, the aggregate supply is $q(p) = \sum_j q_j(p) = \infty$; consequently $q(\cdot)$ is not well defined for this p.



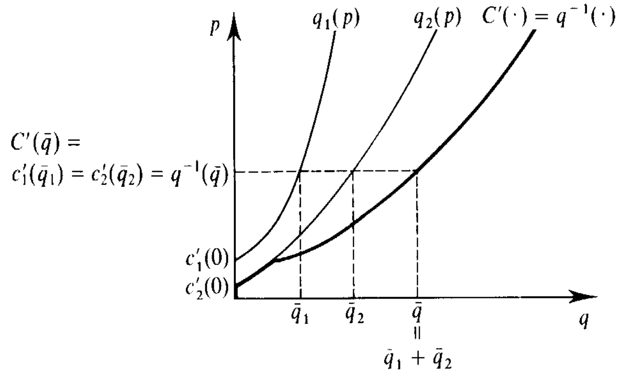


Figure 10.C.4 (left) Equilibrium when $c_j(q_j) = cq_j$ for all $j = 1, \ldots, J$.

Figure 10.C.5 (right) The industry marginal cost function.

basic features of the analysis do not change. Figure 10.C.4 depicts the determination of the equilibrium value of p in the case where, for all j, $c_j(q_j) = cq_j$ for some scalar $c > 0$. The only difference from the strictly convex case is that, when $J > 1$, individual firms' equilibrium production levels are not uniquely determined.

The inverses of the aggregate demand and supply functions also have interpretations that are of interest. At any given level of aggregate output of good ℓ, say $\bar{q}$, the inverse of the industry supply function, $q^{-1}(\bar{q})$, gives the price that brings forth aggregate supply $\bar{q}$. That is, when each firm chooses its optimal output level facing the price $p = q^{-1}(\bar{q})$, aggregate supply is exactly $\bar{q}$. Figure 10.C.5 illustrates this point. Note that in selecting these output levels, all active firms set their marginal cost equal to $q^{-1}(\bar{q})$. As a result, the marginal cost of producing an additional unit of good ℓ at $\bar{q}$ is precisely $q^{-1}(\bar{q})$, regardless of which active firm produces it. Thus $q^{-1}(\cdot)$, the inverse of the industry supply function, can be viewed as the *industry marginal cost function*, which we now denote by $C'(\cdot) = q^{-1}(\cdot)$.[11]

The derivation of $C'(\cdot)$ just given accords fully with our discussion in Section 5.E. We saw there that the aggregate supply of the J firms, $q(p)$, maximizes aggregate profits given p; therefore, we can relate $q(\cdot)$ to the industry marginal cost function $C'(\cdot)$ in exactly the same manner as we did in Section 5.D for the case of a single firm's marginal cost function and supply behavior. With convex technologies, the aggregate supply locus for good ℓ therefore coincides with the graph of the industry marginal cost function $C'(\cdot)$, and so $q^{-1}(\cdot) = C'(\cdot)$.[12]

Likewise, at any given level of aggregate demand $\bar{x}$, the *inverse demand function* $P(\bar{x}) = x^{-1}(\bar{x})$ gives the price that results in aggregate demand of $\bar{x}$. That is, when each consumer optimally chooses his demand for good ℓ at this price, total demand exactly equals $\bar{x}$. Note that at these individual demand levels (assuming that they are positive), each consumer's marginal benefit in terms of the numeraire from an additional unit of good ℓ, $\phi_i'(x_i)$, is exactly equal to $P(\bar{x})$. This is illustrated in Figure

11. Formally, the industry marginal cost function $C'(\cdot)$ is the derivative of the aggregate cost function $C(\cdot)$ that gives the total production cost that would be incurred by a central authority who operates all J firms and seeks to produce any given aggregate level of good ℓ at minimum total cost. (See Exercise 10.C.3.)

12. More formally, by Proposition 5.E.1, aggregate supply behavior can be determined by maximizing profit given the aggregate cost function $C(\cdot)$. This yields first-order condition $p = C'(q(p))$. Hence, $q(\cdot) = C'^{-1}(\cdot)$, or equivalently $q^{-1}(\cdot) = C'(\cdot)$.

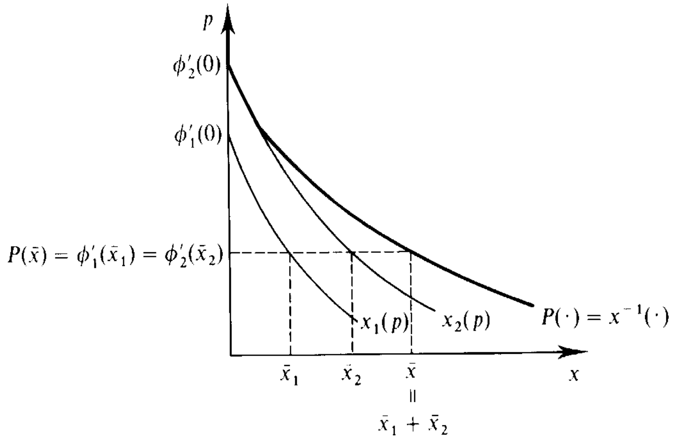


Figure 10.C.6
The inverse demand function.

10.C.6. The value of the inverse demand function at quantity $\bar{x}$, $P(\bar{x})$, can thus be viewed as giving the *marginal social benefit of good* ℓ given that the aggregate quantity $\bar{x}$ is efficiently distributed among the I consumers (see Exercise 10.C.4 for a precise statement of this fact).

Given these interpretations, we can view the competitive equilibrium as involving an aggregate output level at which the marginal social benefit of good ℓ is exactly equal to its marginal cost. This suggests a social optimality property of the competitive allocation, a topic that we investigate further in Section 10.D.

Comparative Statics

It is often of interest to determine how a change in underlying market conditions affects the equilibrium outcome of a competitive market. Such questions may arise, for example, because we may be interested in comparing market outcomes across several similar markets that differ in some measurable way (e.g., we might compare the price of ice cream in a number of cities whose average temperatures differ) or because we want to know how a change in market conditions will alter the outcome in a particular market. The analysis of these sorts of questions is known as *comparative statics analysis.*

As a general matter, we might imagine that each consumer's preferences are affected by a vector of exogenous parameters $\alpha \in \mathbb{R}^M$, so that the utility function $\phi_i(\cdot)$ can be written as $\phi_i(x_i, \alpha)$. Similarly, each firm's technology may be affected by a vector of exogenous parameters $\beta \in \mathbb{R}^S$, so that the cost function $c_j(\cdot)$ can be written as $c_j(q_j, \beta)$. In addition, in some circumstances, consumers and firms face taxes or subsidies that may make the effective (i.e., net of taxes and subsidies) price paid or received differ from the market price p. We let $\hat{p}_i(p, t)$ and $\hat{p}_j(p, t)$ denote, respectively, the effective price paid by consumer i and the effective price received by firm j given tax and subsidy parameters $t \in \mathbb{R}^K$. For example, if consumer i must pay a tax of t_i (in units of the numeraire) per unit of good i purchased, then $\hat{p}_i(p, t) = p + t_i$. If consumer i instead faces a tax that is a percentage t_i of the sales price, then $\hat{p}_i(p, t) = p(1 + t_i)$.

For given values (α, β, t) of the parameters, the $I + J$ equilibrium quantities $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and the equilibrium price p^* are determined as the solution to the following $I + J + 1$ equations (we assume, for simplicity, that $x_i^* > 0$ for all

i and $q_j^* > 0$ for all j):

$$\phi_i'(x_i^*, \alpha) = \hat{p}_i(p^*, t) \qquad i = 1, \ldots, I. \tag{10.C.4}$$

$$c_j'(q_j^*, \beta) = \hat{p}_j(p^*, t) \qquad j = 1, \ldots, J. \tag{10.C.5}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*. \tag{10.C.6}$$

These $I + J + 1$ equations implicitly define the equilibrium allocation and price as functions of the exogenous parameters (α, β, t). If all the relevant functions are differentiable, we can use the implicit function theorem to derive the marginal change in the equilibrium allocation and price in response to a differential change in the values of these parameters (see Section M.E of the Mathematical Appendix). In Example 10.C.1, we consider one such comparative statics exercise; it is only one among a large number of possibilities that arise naturally in economic applications. (The exercises at the end of this chapter include additional examples.)

Example 10.C.1: *Comparative Statics Effects of a Sales Tax.* Suppose that a new sales tax is proposed under which consumers must pay an amount $t \geq 0$ (in units of the numeraire) for each unit of good ℓ consumed. We wish to determine the effect of this tax on the market price. Let $x(p)$ and $q(p)$ denote the aggregate demand and supply functions, respectively, for good ℓ in the absence of the tax (we maintain all our previous assumptions regarding these functions).

In terms of our previous notation, the $\phi_i(\cdot)$ and $c_j(\cdot)$ functions do not depend on any exogenous parameters, $\hat{p}_i(p, t) = p + t$ for all i, and $\hat{p}_j(p, t) = p$ for all j. In principle, by substituting these expressions into the system of equilbrium equations (10.C.4) to (10.C.6), we can derive the effect of a marginal increase in the tax on the price by direct use of the implicit function theorem (see Exercise 10.C.5). Here, however, we pursue a more instructive way to get the answer. In particular, note that aggregate demand with a tax of t and price p is exactly $x(p + t)$ because the tax is equivalent for consumers to the price being increased by t. Thus, the equilibrium market price when the tax is t, which we denote by $p^*(t)$, must satisfy

$$x(p^*(t) + t) = q(p^*(t)). \tag{10.C.7}$$

Suppose that we now want to determine the effect on prices paid and received of a marginal increase in the tax. Assuming that $x(\cdot)$ and $q(\cdot)$ are differentiable at $p = p^*(t)$, differentiating (10.C.7) yields

$$p^{*\prime}(t) = -\frac{x'(p^*(t) + t)}{x'(p^*(t) + t) - q'(p^*(t))}. \tag{10.C.8}$$

It is immediate from (10.C.8) and our assumptions on $x'(\cdot)$ and $q'(\cdot)$ that $-1 \leq p^{*\prime}(t) < 0$ at any t. Therefore, the price $p^*(t)$ received by producers falls as t increases while the overall cost of the good to consumers $p^*(t) + t$ rises (weakly). The total quantities produced and consumed fall (again weakly). See Figure 10.C.7(a), where the equilibrium level of aggregate consumption at tax rate t is denoted by $x^*(t)$. Notice from (10.C.8) that when $q'(p^*(t))$ is large we have $p^{*\prime}(t) \approx 0$, and so the price received by the firms is hardly affected by the tax; nearly all the impact of the tax is felt by consumers. In contrast, when $q'(p^*(t)) = 0$, we have $p^{*\prime}(t) = -1$, and so the impact of the tax is felt entirely by the firms. Figures 10.C.7(b) and (c) depict these two cases.

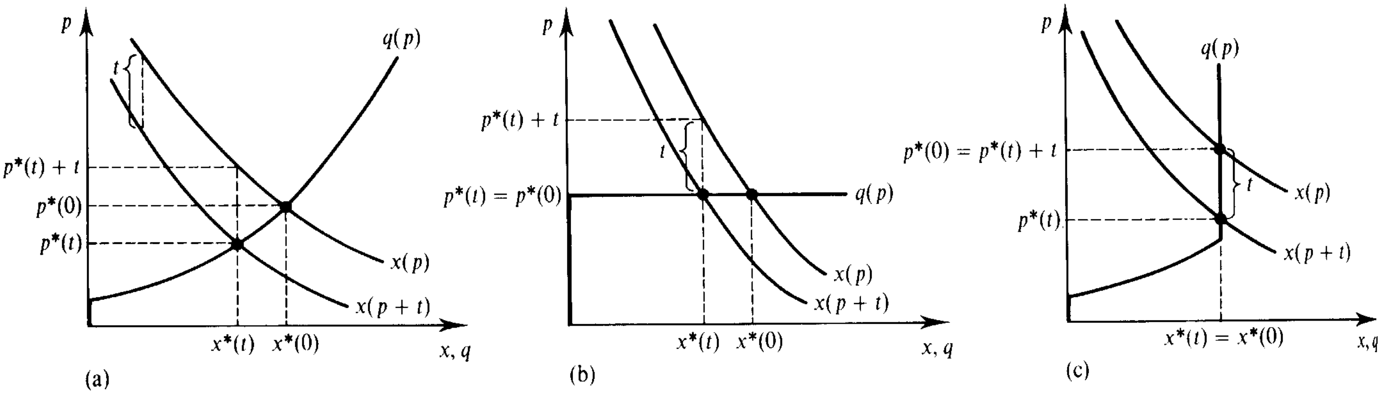


Figure 10.C.7 Comparative statics effects of a sales tax.

By substituting into (10.C.8) for $x'(\cdot)$ and $q'(\cdot)$, the marginal change in p^* can be expressed in terms of derivatives of the underlying individual utility and cost functions. For example, if we let $p^* = p^*(0)$ be the pretax price, we see that

$$p^{*\prime}(0) = -\frac{\sum_{i=1}^{I} [\phi_i''(x_i(p^*))]^{-1}}{\sum_{i=1}^{I} [\phi_i''(x_i(p^*))]^{-1} - \sum_{j=1}^{J} [c_j''(q_j(p^*))]^{-1}}.$$

■

We have assumed throughout this section that consumers' preferences and firms' technologies are convex (and strictly so in the case of consumer preferences). What if this is not the case? Figure 10.C.8 illustrates one problem that can then arise; it shows the demand function and

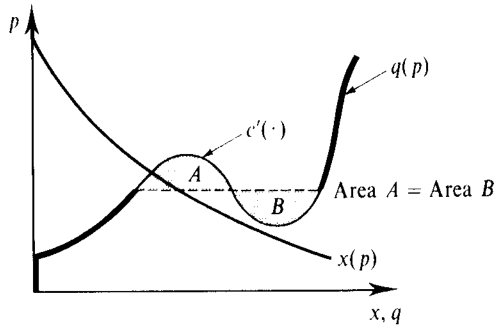


Figure 10.C.8 Nonexistence of competitive equilibrium with a nonconvex technology.

supply correspondence for an economy in which there is a single firm (so $J = 1$).[13] This firm's cost function $c(\cdot)$ is continuous and differentiable but not convex. In the figure, the light curve is the graph of the firm's marginal cost function $c'(\cdot)$. As the figure illustrates, $c'(\cdot)$ fails to be nondecreasing. The heavier curve is the firm's actual supply correspondence $q(\cdot)$ (you should verify that it is determined as indicated in the figure).[14] The graph of the supply correspondence no longer coincides with the marginal cost curve and, as is evident in the figure, no intersection exists between the graph of the supply correspondence and the demand curve. Thus, in this case, *no competitive equilibrium exists.*

13. We set $J = 1$ here solely for expositional purposes.

14. See Section 5.D for a more detailed discussion of the relation between a firm's supply correspondence and its marginal cost function when its technology is nonconvex.

This observation suggests that convexity assumptions are key to the existence of a competitive equilibrium. We shall confirm this in Chapter 17, where we provide a more general discussion of the conditions under which existence of a competitive equilibrium is assured.

10.D The Fundamental Welfare Theorems in a Partial Equilibrium Context

In this section, we study the properties of Pareto optimal allocations in the framework of the two-good quasilinear economy introduced in Section 10.C, and we establish a fundamental link between the set of Pareto optimal allocations and the set of competitive equilibria.

The identification of Pareto optimal allocations is considerably facilitated by the quasilinear specification. In particular, *when consumer preferences are quasilinear, the boundary of the economy's utility possibility set is linear* (see Section 10.B for the definition of this set) *and all points in this boundary are associated with consumption allocations that differ only in the distribution of the numeraire among consumers.*

To see this important fact, suppose that we fix the consumption and production levels of good ℓ at $(\bar{x}_1, \dots, \bar{x}_I, \bar{q}_1, \dots, \bar{q}_J)$. With these production levels, the total amount of the numeraire available for distribution among consumers is $\omega_m - \sum_j c_j(\bar{q}_j)$. Because the quasilinear form of the utility functions allows for an unlimited unit-for-unit transfer of utility across consumers through transfers of the numeraire, the set of utilities that can be attained for the I consumers by appropriately distributing the available amounts of the numeraire is given by

$$\left\{(u_1, \dots, u_I)\colon \sum_{i=1}^{I} u_i \le \sum_{i=1}^{I} \phi_i(\bar{x}_i) + \omega_m - \sum_{j=1}^{J} c_j(\bar{q}_j)\right\}. \tag{10.D.1}$$

The boundary of this set is a hyperplane with normal vector $(1, \dots, 1)$. The set is depicted for the case $I = 2$ by the hatched set in Figure 10.D.1.

Note that by altering the consumption and production levels of good ℓ, we necessarily shift the boundary of this set in a parallel manner. Thus, every Pareto optimal allocation must involve the quantities $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*)$ that extend this boundary as far out as possible, as illustrated by the heavily drawn boundary of the shaded utility possibility set in Figure 10.D.1. We call these quantities the *optimal consumption and production levels* for good ℓ. As long as these optimal consumption and production levels for good ℓ are uniquely determined, Pareto optimal allocations can differ only in the distribution of the numeraire among consumers.[15]

15. The optimal individual production levels need not be unique if firms' cost functions are convex but not strictly so. Indeterminacy of optimal individual production levels arises, for example, when all firms have identical constant returns to scale technologies. However, under our assumptions that the $\phi_i(\cdot)$ functions are strictly concave and that the $c_j(\cdot)$ functions are convex, the optimal individual consumption levels of good ℓ are necessarily unique and, hence, so is the optimal aggregate production level $\sum_j q_j^*$ of good ℓ. This implies that, under our assumptions, the consumption allocations in two different Pareto optimal allocations can differ only in the distribution of numeraire among consumers. If, moreover, the $c_j(\cdot)$ functions are strictly convex, then the optimal individual production levels are also uniquely determined. (See Exercise 10.D.1.)

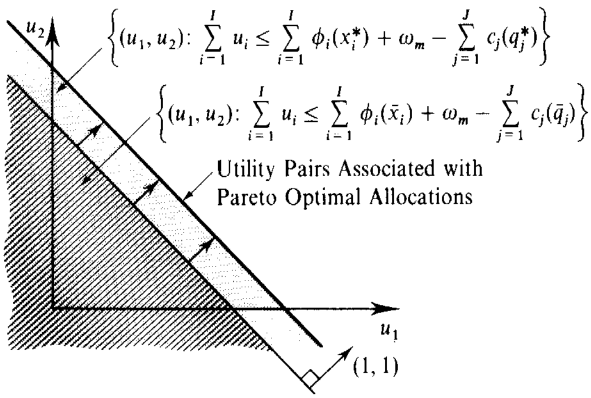


Figure 10.D.1
The utility possibility set in a quasilinear economy.

It follows from expression (10.D.1) that the optimal consumption and production levels of good ℓ can be obtained as the solution to

$$\underset{\substack{(x_1,\ldots,x_I)\geq 0 \\ (q_1,\ldots,q_J)\geq 0}}{\text{Max}} \quad \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) + \omega_m \tag{10.D.2}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} x_i - \sum_{j=1}^{J} q_j = 0.$$

The value of the term $\sum_i \phi_i(x_i) - \sum_j c_j(q_j)$ in the objective function of problem (10.D.2) is known as the *Marshallian aggregate surplus* (or, simply, the *aggregate surplus*). It can be thought of as the total utility generated from consumption of good ℓ less its costs of production (in terms of the numeraire). The optimal consumption and production levels for good ℓ maximize this aggregate surplus measure.

Given our convexity assumptions, the first-order conditions of problem (10.D.2) yield necessary and sufficient conditions that characterize the optimal quantities. If we let μ be the multiplier on the constraint in problem (10.D.2), the $I + J$ optimal values $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ and the multiplier μ satisfy the following $I + J + 1$ conditions:

$$\mu \le c_j'(q_j^*), \quad \text{with equality if } q_j^* > 0 \qquad j = 1, \ldots, J. \tag{10.D.3}$$

$$\phi_i'(x_i^*) \le \mu, \quad \text{with equality if } x_i^* > 0 \qquad i = 1, \ldots, I. \tag{10.D.4}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*. \tag{10.D.5}$$

These conditions should look familiar: They exactly parallel conditions (10.C.1) to (10.C.3) in Section 10.C, with μ replacing p^*. This observation has an important implication. We can immediately infer from it that any competitive equilibrium outcome in this model is Pareto optimal because any competitive equilibrium allocation has consumption and production levels of good ℓ, $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$, that satisfy conditions (10.D.3) to (10.D.5) when we set $\mu = p^*$. Thus, we have established the *first fundamental theorem of welfare economics* (Proposition 10.D.1) in the context of this quasilinear two-good model.

Proposition 10.D.1: (*The First Fundamental Theorem of Welfare Economics*) If the price p^* and allocation $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ constitute a competitive equilibrium, then this allocation is Pareto optimal.

The first fundamental welfare theorem establishes conditions under which market equilibria are necessarily Pareto optimal. It is a formal expression of Adam Smith's "invisible hand" and is a result that holds with considerable generality (see Section 16.C for a much more extensive discussion). Equally important, however, are the conditions under which it fails to hold. In the models for which we establish the first fundamental welfare theorem here and in Section 16.C, markets are "complete" in the sense that there is a market for every relevant commodity and all market participants act as price takers. In Chapters 11 to 14, we study situations in which at least one of these conditions fails, and market outcomes fail to be Pareto optimal as a result.

We can also develop a converse to Proposition 10.D.1, known as the *second fundamental theorem of welfare economics*. In Section 10.C, we saw that good ℓ's equilibrium price p^*, its equilibrium consumption and production levels $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$, and firms' profits are unaffected by changes in consumers' wealth levels. As a result, a transfer of one unit of the numeraire from consumer i to consumer i' will cause each of these consumers' equilibrium consumption of the numeraire to change by exactly the amount of the transfer and will cause no other changes. Thus, by appropriately transferring endowments of the numeraire commodity, the resulting competitive equilibrium allocation can be made to yield any utility vector in the boundary of the utility possibility set. The second welfare theorem therefore tells us that, in this two-good quasilinear economy, a central authority interested in achieving a particular Pareto optimal allocation can always implement this outcome by transferring the numeraire among consumers and then "allowing the market to work." This is stated formally in Proposition 10.D.2.

Proposition 10.D.2: (*The Second Fundamental Theorem of Welfare Economics*) For any Pareto optimal levels of utility $(u_1^*, \ldots, u_I^*)$, there are transfers of the numeraire commodity $(T_1, \ldots, T_I)$ satisfying $\sum_i T_i = 0$, such that a competitive equilibrium reached from the endowments $(\omega_{m1} + T_1, \ldots, \omega_{mI} + T_I)$ yields precisely the utilities $(u_1^*, \ldots, u_I^*)$.

In Section 16.D, we study the conditions under which the second welfare theorem holds in more general competitive economies. A critical requirement, in addition to those needed for the first welfare theorem, turns out to be convexity of preferences and production sets, an assumption we have made in the model under consideration here. In contrast, we shall see in Chapter 16 that no such convexity assumptions are needed for the first welfare theorem.

The correspondence between p and μ in the equilibrium conditions (10.C.1) to (10.C.3) and the Pareto optimality conditions (10.D.3) to (10.D.5) is worthy of emphasis: The competitive price is exactly equal to the shadow price on the resource constraint for good ℓ in the Pareto optimality problem (10.D.2). In this sense, then, we can say that a good's price in a competitive equilibrium reflects precisely its marginal social value. In a competitive equilibrium, each firm, by operating at a point where price equals marginal cost, equates its marginal production cost to the marginal social value of its output. Similarly, each consumer, by consuming up to the point where marginal utility from a good equals its price, is at a point where the marginal benefit from consumption of the good exactly equals its marginal cost. This correspondence between equilibrium market prices and optimal shadow prices holds quite generally in competitive economies (see Section 16.F for further discussion of this point).

An alternative way to characterize the set of Pareto optimal allocations is to solve

$$\begin{aligned} \underset{\{x_i, m_i\}_{i=1}^I, \{z_j, q_j\}_{j=1}^J}{\text{Max}} \quad & m_1 + \phi_1(x_1) \\ \text{s.t.} \quad (1) \quad & m_i + \phi_i(x_i) \geq \bar{u}_i \qquad i = 2, \dots, I \\ (2\ell) \quad & \sum_{i=1}^I x_i - \sum_{j=1}^J q_j \leq 0 \\ (2m) \quad & \sum_{i=1}^I m_i + \sum_{j=1}^J z_j \leq \omega_m \\ (3) \quad & z_j \geq c_j(q_j) \qquad j = 1, \dots, J. \end{aligned} \tag{10.D.6}$$

Problem (10.D.6) expresses the Pareto optimality problem as one of trying to maximize the well-being of individual 1 subject to meeting certain required utility levels for the other individuals in the economy [constraints (1)], resource constraints [constraints (2ℓ) and ($2m$)], and technological constraints [constraints (3)]. By solving problem (10.D.6) for various required levels of utility for these other individuals, $(\bar{u}_2, \dots, \bar{u}_I)$, we can identify all the Pareto optimal outcomes for this economy (see Exercise 10.D.3; more generally, we can do this whenever consumer preferences are strongly monotone). Exercise 10.D.4 asks you to derive conditions (10.D.3) to (10.D.5) in this alternative manner.

10.E Welfare Analysis in the Partial Equilibrium Model

It is often of interest to measure the change in the level of social welfare that would be generated by a change in market conditions such as an improvement in technology, a new government tax policy, or the elimination of some existing market imperfection. In the partial equilibrium model, it is particularly simple to carry out this welfare analysis. This fact accounts to a large extent for the popularity of the model.

In the discussion that follows, we assume that the welfare judgments of society are embodied in a social welfare function $W(u_1, \dots, u_I)$ assigning a social welfare value to every utility vector $(u_1, \dots, u_I)$ (see Chapters 4, 16, and 22 for more on this concept). In addition, we suppose that (as in the theory of the normative representative consumer discussed in Section 4.D) there is some central authority who redistributes wealth by means of transfers of the numeraire commodity in order to maximize social welfare.[16] The critical simplification offered by the quasilinear specification of individual utility functions is that when there is a central authority who redistributes wealth in this manner, *changes in social welfare can be measured by changes in the Marshallian aggregate surplus* (introduced in Section 10.D) *for any social welfare function that society may have.*

To see this point (which we have in fact already examined in Example 4.D.2), consider some given consumption and production levels of good ℓ, $(x_1, \dots, x_I, q_1, \dots, q_J)$,

16. As in Section 4.D, we assume that consumers treat these transfers as independent of their own actions; that is, in the standard terminology, they are *lump-sum* transfers. You should think of the central authority as making the transfers prior to the opening of markets.

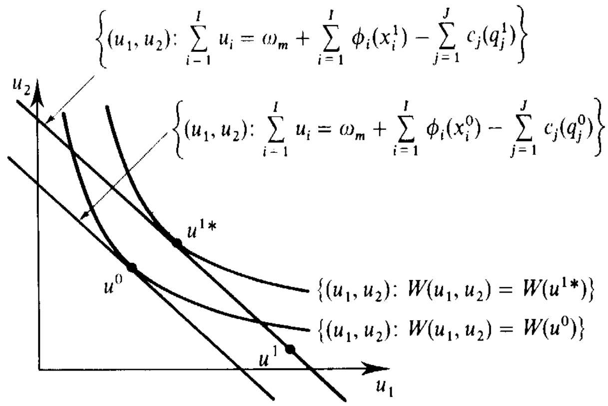


Figure 10.E.1 With lump-sum redistribution occurring to maximize social welfare, changes in welfare correspond to changes in aggregate surplus in a quasilinear model.

having $\sum_i x_i = \sum_j q_j$. From Section 10.D and Figure 10.D.1 we know that the utility vectors $(u_1, \dots, u_I)$ that are achievable through reallocation of the numeraire given these consumption and production levels of good ℓ are

$$\left\{(u_1, \dots, u_I)\colon \sum_{i=1}^{I} u_i \le \omega_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)\right\}.$$

Now, if a central authority is redistributing the numeraire to maximize $W(u_1, \dots, u_I)$, the ultimate maximized value of welfare must be greater the larger this set is (i.e., the farther out the boundary of the set is). Hence, we see that a change in the consumption and production levels of good ℓ leads to an increase in welfare (given optimal redistribution of the numeraire) if and only if it increases the Marshallian aggregate surplus

$$S(x_1, \dots, x_I, q_1, \dots, q_J) = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j). \tag{10.E.1}$$

Figure 10.E.1 provides an illustration. It shows three utility vectors for the case $I = 2$: An initial utility vector $u^0 = (u_1^0, u_2^0)$ associated with an allocation in which the consumption and production levels of good ℓ are $(x_1^0, \dots, x_I^0, q_1^0, \dots, q_J^0)$ and in which the wealth distribution has been optimized, a utility vector $u^1 = (u_1^1, u_2^1)$ that results from a change in the consumption and production levels of good ℓ to $(x_1^1, \dots, x_I^1, q_1^1, \dots, q_J^1)$ in the absence of any transfers of the numeraire, and a utility vector $u^{1*} = (u_1^{1*}, u_2^{1*})$ that results from this change once redistribution of the numeraire occurs to optimize social welfare. As can be seen in the figure, the change increases aggregate surplus and also increases welfare once optimal transfers of the numeraire occur, even though welfare would decrease in the absence of the transfers. Thus, as long as redistribution of wealth is occurring to maximize a social welfare function, changes in welfare can be measured by changes in Marshallian aggregate surplus (to repeat: for *any* social welfare function).[17]

In many circumstances of interest, the Marshallian surplus has a convenient and

17. Notice that no transfers would be necessary in the special case in which the social welfare function is in fact the "utilitarian" social welfare function $\sum_i u_i$; in this case, it is sufficient that all available units of the numeraire go to consumers (i.e., none goes to waste or is otherwise withheld).

historically important formulation in terms of areas lying vertically between the aggregate demand and supply functions for good ℓ.

To expand on this point, we begin by making two key assumptions. Denoting by $x = \sum_i x_i$ the aggregate consumption of good ℓ, we assume, first, that for any x, the individual consumptions of good ℓ are distributed optimally across consumers. That is, recalling our discussion of the inverse demand function $P(\cdot)$ in Section 10.C (see Figure 10.C.6), that we have $\phi_i'(x_i) = P(x)$ for every i. This condition will be satisfied if, for example, consumers act as price-takers and all consumers face the same price. Similarly, denoting by $q = \sum_j q_j$ the aggregate output of good ℓ, we assume that the production of any total amount q is distributed optimally across firms. That is, recalling our discussion of the industry marginal cost curve $C'(\cdot)$ in Section 10.C (see Figure 10.C.5), that we have $c_j'(q_j) = C'(q)$ for every j. This will be satisfied if, for example, firms act as price takers and all firms face the same price. Observe that we do not require that the price faced by consumers and firms be the same.[18]

Consider now a differential change $(dx_1, \ldots, dx_I, dq_1, \ldots, dq_J)$ in the quantities of good ℓ consumed and produced satisfying $\sum_i dx_i = \sum_j dq_j$, and denote $dx = \sum_i dx_i$. The change in aggregate Marshallian surplus is then

$$dS = \sum_{i=1}^{I} \phi_i'(x_i)\, dx_i - \sum_{j=1}^{J} c_j'(q_j)\, dq_j. \tag{10.E.2}$$

Since $\phi_i'(x_i) = P(x)$ for all i, and $c_j'(q_j) = C'(q)$ for all j, we get

$$dS = P(x) \sum_{i=1}^{I} dx_i - C'(q) \sum_{j=1}^{J} dq_j. \tag{10.E.3}$$

Finally, since $x = q$ (by market feasibility) and $\sum_j dq_j = \sum_i dx_i = dx$, this becomes

$$dS = [P(x) - C'(x)]\, dx. \tag{10.E.4}$$

This differential change in Marshallian surplus is depicted in Figure 10.E.2(a). Expression (10.E.4) is quite intuitive; it tells us that starting at aggregate consumption level x the marginal effect on social welfare of an increase in the aggregate quantity consumed, dx, is equal to consumers' marginal benefit from this consumption, $P(x)\, dx$, less the marginal cost of this extra production, $C'(x)\, dx$ (both in terms of the numeraire).

We can also integrate (10.E.4) to express the total value of the aggregate Marshallian surplus at the aggregate consumption level x, denoted $S(x)$, in terms of an integral of the difference between the inverse demand function and the industry marginal cost function,

$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]\, ds, \tag{10.E.5}$$

18. For example, consumers may face a tax per unit purchased that makes the price they pay differ from the price received by the firms (see Example 10.C.1). The assumptions made here also hold in the monopoly model to be studied in Section 12.B. In that model, there is a single firm (and so there is no issue of optimal allocation of production), and all consumers act as price takers facing the same price. An example where the assumption of an optimal allocation of production is not valid is the Cournot duopoly model of Chapter 12 when firms have different efficiencies. There, firms with different costs have different levels of marginal cost in an equilibrium.

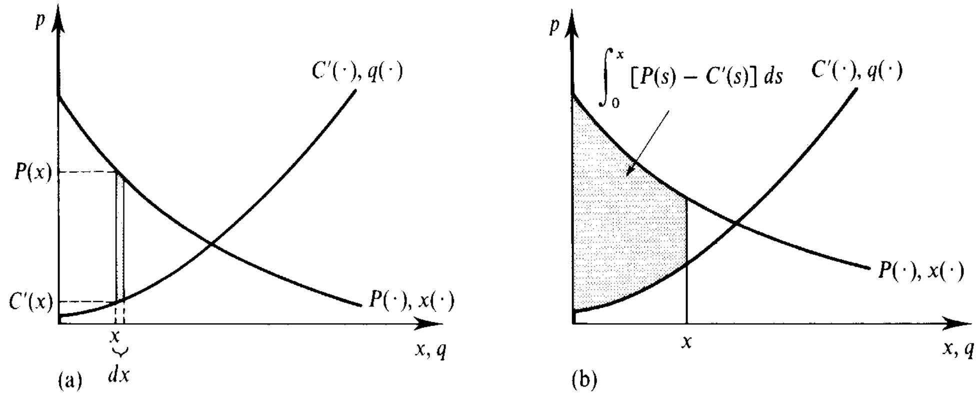


Figure 10.E.2
(a) A differential change in Marshallian surplus. (b) The Marshallian surplus at aggregate consumption level x.

where S_0 is a constant of integration equal to the value of the aggregate surplus when there is no consumption or production of good ℓ [it is equal to zero if $c_j(0) = 0$ for all j]. The integral in (10.E.5) is depicted in Figure 10.E.2(b); it is exactly equal to the area lying vertically between the aggregate demand and supply curves for good ℓ up to quantity x.

Note from (10.E.5) that the value of the aggregate Marshallian surplus is maximized at the aggregate consumption level x^* such that $P(x^*) = C'(x^*)$, which is exactly the competitive equilibrium aggregate consumption level.[19] This accords with Proposition 10.D.1, the first fundamental welfare theorem, which states that the competitive allocation is Pareto optimal.

Example 10.E.1: *The Welfare Effects of a Distortionary Tax.* Consider again the commodity tax problem studied in Example 10.C.1. Suppose now that the welfare authority keeps a balanced budget and returns the tax revenue raised to consumers by means of lump-sum transfers. What impact does this tax-and-transfer scheme have on welfare?[20]

To answer this question, it is convenient to let $(x_1^*(t), \ldots, x_I^*(t), q_1^*(t), \ldots, q_J^*(t))$ and $p^*(t)$ denote the equilibrium consumption, production, and price levels of good ℓ when the tax rate is t. Note that $\phi_i'(x_i^*(t)) = p^*(t) + t$ for all i and that $c_j'(q_j^*(t)) = p^*(t)$ for all j. Thus, letting $x^*(t) = \sum_i x_i^*(t)$ and $S^*(t) = S(x^*(t))$, we can use (10.E.5) to express the change in aggregate Marshallian surplus resulting from

19. To see this, check first that $S''(x) \leq 0$ at all x. Hence, $S(\cdot)$ is a concave function and therefore $x^* > 0$ maximizes aggregate surplus if and only if $S'(x^*) = 0$. Then verify that $S'(x) = P(x) - C'(x)$ at all $x > 0$.

20. This problem is closely related to that studied in Example 3.I.1 (we could equally well motivate the analysis here by asking, as we did there, about the welfare cost of the distortionary tax relative to the use of a lump-sum tax that raises the same revenue; the measure of deadweight loss that emerges would be the same as that developed here). The discussion that follows amounts to an extension, in the quasilinear context, of the analysis of Example 3.I.1 to situations with many consumers and the presence of firms. For an approach that uses the theory of a normative representative consumer presented in Section 4.D, see the small-type discussion at the end of this section.

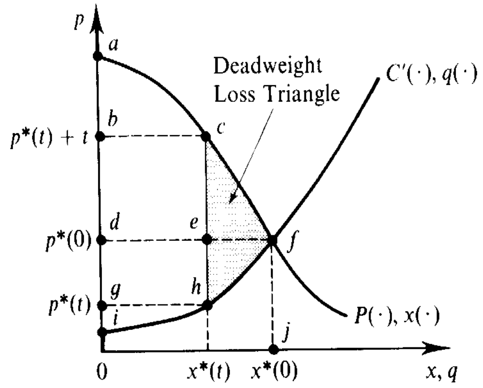


Figure 10.E.3 The deadweight welfare loss from distortionary taxation.

the introduction of the tax as

$$S^*(t) - S^*(0) = \int_{x^*(0)}^{x^*(t)} [P(s) - C'(s)]\, ds. \tag{10.E.6}$$

Expression (10.E.6) is negative because $x^*(t) < x^*(0)$ (recall the analysis of Example 10.C.1) and $P(x) \geq C'(x)$ for all $x \leq x^*(0)$, with strict inequality for $x < x^*(0)$. Hence, social welfare is optimized by setting $t = 0$. The loss in welfare from $t > 0$ is known as the *deadweight loss of distortionary taxation* and is equal to the area of the shaded region in Figure 10.E.3, called the *deadweight loss triangle*.

Notice that since $S^{*\prime}(t) = [P(x^*(t)) - C'(x^*(t))]x^{*\prime}(t)$, we have $S^{*\prime}(0) = 0$. That is, starting from a position without any tax, the first-order welfare effect of an infinitesimal tax is zero. Only as the tax rate increases above zero does the marginal effect become strictly negative. This is as it should be: if we start at an (interior) welfare maximum, then a small displacement from the optimum cannot have a first-order effect on welfare.

It is sometimes of interest to distinguish between the various components of aggregate Marshallian surplus that accrue directly to consumers, firms, and the tax authority.[21] The *aggregate consumer surplus* when consumers' effective price is $\hat{p}$ and therefore aggregate consumption is $x(\hat{p})$ is defined as the gross consumer benefits from consumption of good ℓ minus the consumers' total expenditure on this good (the latter is the cost to consumers in terms of forgone consumption of the numeraire):

$$CS(\hat{p}) = \sum_{i=1}^{I} \phi_i(x_i(\hat{p})) - \hat{p}x(\hat{p}).$$

Using again the fact that consumption is distributed optimally, we have

$$\begin{aligned} CS(\hat{p}) &= \int_0^{x(\hat{p})} P(s)\, ds - \hat{p}x(\hat{p}) \\ &= \int_0^{x(\hat{p})} [P(s) - \hat{p}]\, ds. \end{aligned} \tag{10.E.7}$$

21. For example, if the set of active consumers of good ℓ is distinct from the set of owners of the firms producing the good, then this distinction tells us something about the distributional effects of the tax in the absence of transfers between owners and consumers.

Finally, the integral in (10.E.7) is equal to[22]

$$CS(\hat{p}) = \int_{\hat{p}}^{\infty} x(s)\, ds. \tag{10.E.8}$$

Thus, because consumers face an effective price of $p^*(t) + t$ when the tax is t, the change in consumer surplus from imposition of the tax is

$$CS(p^*(t) + t) - CS(p^*(0)) = -\int_{p^*(0)}^{p^*(t)+t} x(s)\, ds. \tag{10.E.9}$$

In Figure 10.E.3, the reduction in consumer surplus is depicted by area $(dbcf)$.

The aggregate profit, or *aggregate producer surplus*, when firms face effective price $\hat{p}$ is

$$\Pi(\hat{p}) = \hat{p}q(\hat{p}) - \sum_{j=1}^{J} c_j(q_j(\hat{p})).$$

Again, using the optimality of the allocation of production across firms, we have[23]

$$\Pi(\hat{p}) = \Pi_0 + \int_0^{q(\hat{p})} [\hat{p} - C'(s)]\, ds \tag{10.E.10}$$

$$= \Pi_0 + \int_0^{\hat{p}} q(s)\, ds, \tag{10.E.11}$$

where Π_0 is a constant of integration equal to profits when $q_j = 0$ for all j [$\Pi_0 = 0$ if $c_j(0) = 0$ for all j]. Since producers pay no tax, they face price $p^*(t)$ when the tax rate is t. The change in producer surplus is therefore

$$\Pi(p^*(t)) - \Pi(p^*(0)) = -\int_{p^*(t)}^{p^*(0)} q(s)\, ds. \tag{10.E.12}$$

The reduction in producer surplus is depicted by area $(gdfh)$ in Figure 10.E.3.

Finally, the *tax revenue* is $tx^*(t)$; it is depicted in Figure 10.E.3 by area $(gbch)$.

The total deadweight welfare loss from the tax is then equal to the sum of the reductions in consumer and producer surplus less the tax revenue. ■

The welfare measure developed here is closely related to our discussion of normative representative consumers in Section 4.D. We showed there that if a central authority is redistributing wealth to maximize a social welfare function given prices p, leading to a wealth distribution rule $(w_1(p, w), \ldots, w_I(p, w))$, then there is a normative representative consumer with indirect utility function $v(p, w)$ whose demand $x(p, w)$ is exactly equal to aggregate demand [i.e., $x(p, w) = \sum_i x_i(p, w_i(p, w))$] and whose utility can be used as a measure of social welfare. Recalling our discussion in Section 3.I, this means that we can measure the change in welfare resulting from a price–wealth change by adding the representative consumer's

22. This can be seen geometrically. For example, when $\hat{p} = p^*(0)$, the integrals in both (10.E.7) and (10.E.8) are equal to area (daf) in Figure 10.E.3. Formally, the equivalence follows from a change of variables and integration by parts (see Exercise 10.E.2).

23. When $\hat{p} = p^*(0)$, the integrals in both (10.E.10) and (10.E.11) are equal to area (idf) in Figure 10.E.3. The equivalence of these two integrals again follows formally by a change of variables and integration by parts.

compensating or equivalent variation for the price change to the change in the representative consumer's wealth (see Exercise 3.I.12). But in the quasilinear case, the representative consumer's compensating and equivalent variations are the same and can be calculated by direct integration of the representative consumer's Walrasian demand function, that is, by integration of the aggregate demand function. Hence, in Example 10.E.1, the representative consumer's compensating variation for the price change is exactly equal to the change in aggregate consumer surplus, expression (10.E.9). The change in the representative consumer's wealth, on the other hand, is equal to the change in aggregate profits plus the tax revenue rebated to consumers. Thus, the total welfare change arising from the introduction of the tax-and-transfer scheme, as measured using the normative representative consumer, is exactly equal to the deadweight loss calculated in Example 10.E.1.[24]

Another way to justify the use of aggregate surplus as a welfare measure in the quasilinear model is as a measure of *potential Pareto improvement*. Consider the tax example. We could say that a change in the tax represents a *potential* Pareto improvement if there is a set of lump-sum transfers of the numeraire that would make all consumers better off than they were before the tax change. In the present quasilinear context, this is true if and only if aggregate surplus increases with the change in the tax. This approach is sometimes referred to as the *compensation principle* because it asks whether, in principle, it is possible given the change for the winners to compensate the losers so that all are better off than before. (See also the discussion in Example 4.D.2 and especially Section 22.C.)

We conclude this section with a warning: When the numeraire represents many goods, the welfare analysis we have performed is justified only if the prices of goods other than good ℓ are undistorted in the sense that they equal these goods' true marginal utilities and production costs. Hence, these other markets must be competitive, and all market participants must face the same price. If this condition does not hold, then the costs of production faced by producers of good ℓ do not reflect the true social costs incurred from their use of these goods as inputs. Exercise 10.G.3 provides an illustration of this problem.

10.F Free-Entry and Long-Run Competitive Equilibria

Up to this point, we have taken the set of firms and their technological capabilities as fixed. In this section, we consider the case in which an infinite number of firms can potentially be formed, each with access to the most efficient production technology. Moreover, firms may enter or exit the market in response to profit opportunities. This scenario, known as a situation of *free entry*, is often a reasonable approximation when we think of long-run outcomes in a market. In the discussion that follows, we introduce and study a notion of *long-run competitive equilibrium* and then discuss how this concept can be used to analyze long-run and short-run comparative statics effects.

To begin, suppose that each of an infinite number of potential firms has access to a technology for producing good ℓ with cost function $c(q)$, where q is the *individual* firm's output of good ℓ. We assume that $c(0) = 0$; that is, a firm can earn zero profits by simply deciding to be inactive and setting $q = 0$. In the terminology of Section

24. This deadweight loss measure corresponds also to the measure developed for the one-consumer case in Example 3.I.1, where we implicitly limited ourselves to the case in which the taxed good has a constant unit cost.

5.B, there are no sunk costs in the long run. The aggregate demand function is $x(\cdot)$, with inverse demand function $P(\cdot)$.

In a long-run competitive equilibrium, we would like to determine not only the price and output levels for the firms but also the number of firms that are active in the industry. Given our assumption of identical firms, we focus on equilibria in which all active firms produce the same output level, so that a long-run competitive equilibrium can be described by a triple (p, q, J) formed by a price p, an output per firm q, and an integer number of active firms J (hence the total industry output is $Q = Jq$).[25] The central assumption determining the number of active firms is one of free entry and exit: A firm will enter the market if it can earn positive profits at the going market price and will exit if it can make only negative profits at any positive production level given this price. If all firms, active and potential, take prices as unaffected by their own actions, this implies that active firms must earn exactly zero profits in any long-run competitive equilibrium; otherwise, we would have either no firms willing to be active in the market (if profits were negative) or an infinite number of firms entering the market (if profits were positive). This leads us to the formulation given in Definition 10.F.1.

Definition 10.F.1: Given an aggregate demand function $x(p)$ and a cost function $c(q)$ for each potentially active firm having $c(0) = 0$, a triple (p^*, q^*, J^*) is a *long-run competitive equilibrium* if

(i) q^* solves $\underset{q \geq 0}{\text{Max}}\; p^*q - c(q)$ (Profit maximization)

(ii) $x(p^*) = J^*q^*$ (Demand = supply)

(iii) $p^*q^* - c(q^*) = 0$ (Free Entry Condition).

The long-run equilibrium price can be thought of as equating demand with long-run supply, where the long-run supply takes into account firms' entry and exit decisions. In particular, if $q(\cdot)$ is the supply correspondence of an individual firm with cost function $c(\cdot)$ and $\pi(\cdot)$ is its profit function, we can define a *long-run aggregate supply correspondence* by[26]

$$Q(p) = \begin{cases} \infty & \text{if } \pi(p) > 0, \\ \{Q \geq 0 \colon Q = Jq \text{ for some integer } J \geq 0 \text{ and } q \in q(p)\} & \text{if } \pi(p) = 0. \end{cases}$$

If $\pi(p) > 0$, then every firm wants to supply an amount strictly bounded away from zero. Hence, the aggregate supply is infinite. If $\pi(p) = 0$ and $Q = Jq$ for some $q \in q(p)$, then we can have J firms each supply q and have the rest remain inactive [since $c(0) = 0$, this is a profit-maximizing choice for the inactive firms as well]. With this

25. The assumption that all active firms produce the same output level is without loss of generality whenever $c(\cdot)$ is strictly convex on the set $(0, \infty]$. A firm's supply correspondence can then include at most one positive output level at any given price p.

26. In terms of the basic properties of production sets presented in Section 5.B, the long-run supply correspondence is the supply correspondence of the production set Y^+, where Y is the production set associated with the individual firm [i.e., with $c(\cdot)$], and Y^+ is its "additive closure" (i.e., the smallest set that contains Y and is additive: $Y^+ + Y^+ \subset Y^+$; see Exercise 5.B.4).

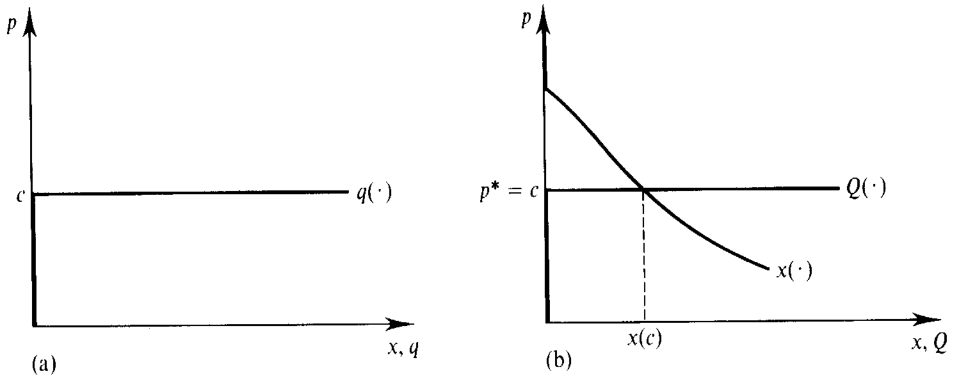


Figure 10.F.1
Long-run competitive equilibrium with constant returns to scale. (a) A firm's supply correspondence. (b) Long-run equilibrium.

notion of a long-run supply correspondence, p^* is a long-run competitive equilibrium price if and only if $x(p^*) \in Q(p^*)$.[27]

We now investigate this long-run competitive equilibrium notion. Consider first the case in which the cost function $c(\cdot)$ exhibits constant returns to scale, so that $c(q) = cq$ for some $c > 0$, and assume that $x(c) > 0$. In this case, condition (i) of Definition 10.F.1 tells us that in any long-run competitive equilbrium we have $p^* \leq c$ (otherwise, there is no profit-maximizing production). However, at any such price, aggregate consumption is strictly positive since $x(c) > 0$, so condition (ii) requires that $q^* > 0$, By condition (iii), we must have $(p^* - c)q^* = 0$. Hence, we conclude that $p^* = c$ and aggregate consumption is $x(c)$. Note, however, that J^* and q^* are *indeterminate*: any J^* and q^* such that $J^*q^* = x(c)$ satisfies conditions (i) and (ii).

Figure 10.F.1 depicts this long-run equilibrium. The supply correspondence of an individual firm $q(\cdot)$ is illustrated in Figure 10.F.1(a); Figure 10.F.1(b) shows the long-run equilibrium price and aggregate output as the intersection of the graph of the aggregate demand function $x(\cdot)$ with the graph of the long-run aggregate supply correspondence

$$Q(p) = \begin{cases} \infty & \text{if } p > c \\ [0, \infty) & \text{if } p = c \\ 0 & \text{if } p < c. \end{cases}$$

We move next to the case in which $c(\cdot)$ is increasing and strictly convex (i.e., the production technology of an individual firm displays strictly decreasing returns to scale). We assume also that $x(c'(0)) > 0$. With this type of cost function, *no long-run competitive equilibrium can exist*. To see why this is so, note that if $p > c'(0)$, then $\pi(p) > 0$ and therefore the long-run supply is infinite. On the other hand, if $p \leq c'(0)$, then the long-run supply is zero while $x(p) > 0$. The problem is illustrated in Figure 10.F.2, where the graph of the demand function $x(\cdot)$ has no intersection with the

27. In particular, if (p^*, q^*, J^*) is a long-run equilibrium, then condition (i) of Definition 10.F.1 implies that $q^* \in q(p^*)$ and condition (iii) implies that $\pi(p^*) = 0$. Hence, by condition (ii), $x(p^*) \in Q(p^*)$. In the other direction, if $x(p^*) \in Q(p^*)$, then $\pi(p^*) = 0$ and there exists $q^* \in q(p^*)$ and J^* with $x(p^*) = J^*q^*$. Therefore, the three conditions of Definition 10.F.1 are satisfied.

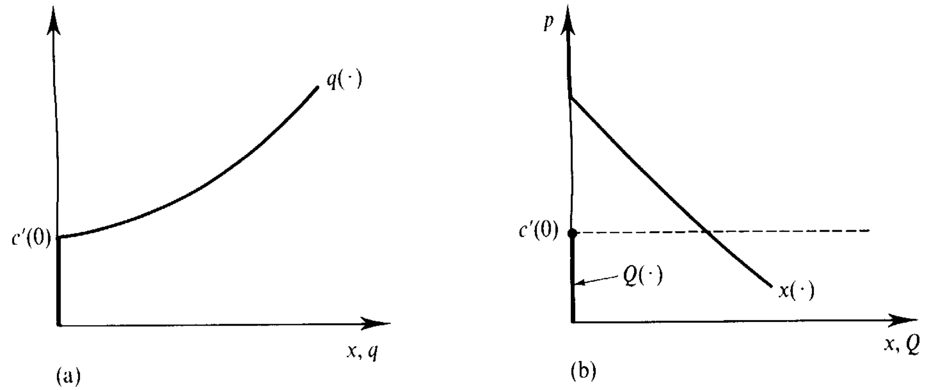


Figure 10.F.2
Nonexistence of long-run competitive equilibrium with strictly convex costs. (a) A firm's supply correspondence. (b) No intersection of long-run supply and demand.

graph of the long-run aggregate supply correspondence

$$Q(p) = \begin{cases} \infty & \text{if } p > c'(0) \\ 0 & \text{if } p \le c'(0). \end{cases}$$

The difficulty can be understood in a related way. As discussed in Exercise 5.B.4, the long-run aggregate production set in the situation just described is convex but not closed. This can be seen in Figure 10.F.3, where the industry marginal cost function with J firms,

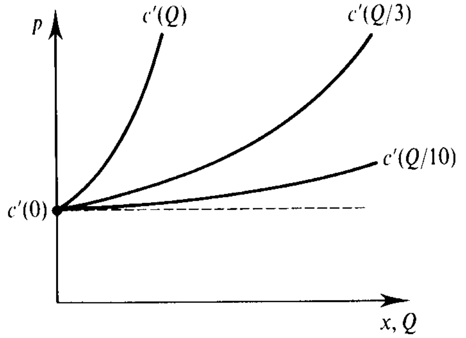


Figure 10.F.3
The limiting behavior of industry marginal cost as $J \to \infty$ with strictly convex costs.

$c'(Q/J)$, is shown for various values of J (in particular, for $J = 1$, $J = 3$, and $J = 10$). Note that as J increases, this marginal cost function approaches *but never reaches* the marginal cost function corresponding to a constant marginal cost of $c'(0)$.

Perhaps not surprisingly, to generate the existence of an equilibrium with a determinate number of firms, the long-run cost function must exhibit a strictly positive efficient scale; that is, *there must exist a strictly positive output level $\bar{q}$ at which a firm's average costs of production are minimized* (see Section 5.D for a further discussion of the efficient scale concept).

Suppose, in particular, that $c(\cdot)$ has a unique efficient scale $\bar{q} > 0$, and let the minimized level of average cost be $\bar{c} = c(\bar{q})/\bar{q}$. Assume, moreover, that $x(\bar{c}) > 0$. If at a long-run equilibrium (p^*, q^*, J^*) we had $p^* > \bar{c}$, then $p^*\bar{q} > \bar{c}\bar{q}$, and so we would have $\pi(p^*) > 0$. Thus, at any long-run equilibrium we must have $p^* \le \bar{c}$. In contrast, if $p^* < \bar{c}$, then $x(p^*) > 0$; but since $p^*q - c(q) = p^*q - (c(q)/q)q \le (p^* - \bar{c})q < 0$

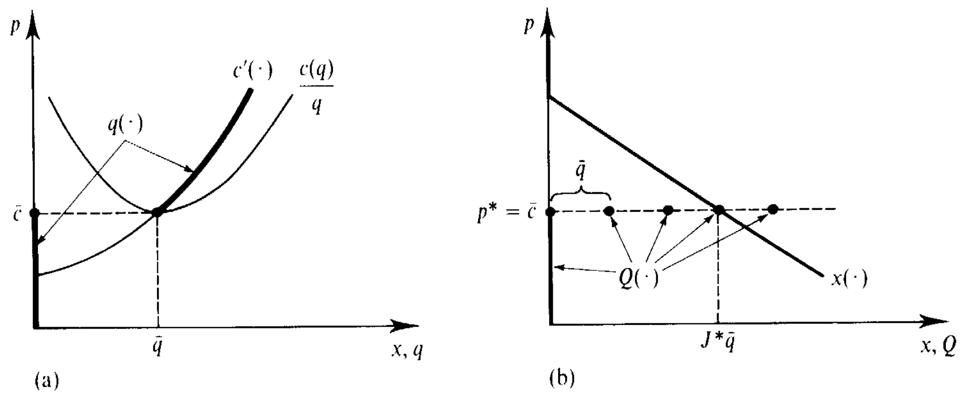


Figure 10.F.4 Long-run competitive equilibrium when average costs exhibit a strictly positive efficient scale. (a) A firm's supply correspondence. (b) Long-run equilibrium.

for all $q > 0$, a firm would earn strictly negative profits at any positive level of output. So $p^* < \bar{c}$ also cannot be a long-run equilibrium price. Thus, at any long-run equilibrium we must have $p^* = \bar{c}$. Moreover, if $p^* = \bar{c}$, then each active firm's supply must be $q^* = \bar{q}$ (this is the only strictly positive output level at which the firm earns nonnegative profits), and the equilibrium number of active firms is therefore $J^* = x(\bar{c})/\bar{q}$.[28] In conclusion, the number of active firms is a well-determined quantity at long-run equilibrium. Figure 10.F.4 depicts such an equilibrium. The long-run aggregate supply correspondence is

$$Q(p) = \begin{cases} \infty & \text{if } p > \bar{c} \\ \{Q \geq 0 \colon Q = J\bar{q} \text{ for some integer } J \geq 0\} & \text{if } p = \bar{c} \\ 0 & \text{if } p < \bar{c}. \end{cases}$$

Observe that the equilibrium price and aggregate output are exactly the same as if the firms had a constant returns to scale technology with unit cost $\bar{c}$.

Several points should be noted about the equilibrium depicted in Figure 10.F.4. First, if the efficient scale of operation is large relative to the size of market demand, it could well turn out that the equilibrium number of active firms is small. In these cases, we may reasonably question the appropriateness of the price-taking assumption (e.g., what if $J^* = 1$?). Indeed, we are then likely to be in the realm of the situations with market power studied in Chapter 12.

Second, we have conveniently shown the demand at price $\bar{c}$, $x(\bar{c})$, to be an integer multiple of $\bar{q}$. Were this not so, no long-run equilibrium would exist because the graphs of the demand function and the long-run supply correspondence would

28. Note that when $c(\cdot)$ is differentiable, condition (i) of Definition 10.F.1 implies that $c'(q^*) = p^*$, while condition (iii) implies $p^* = c(q^*)/q^*$. Thus, a *necessary* condition for an equilibrium is that $c'(q^*) = c(q^*)/q^*$. This is the condition for q^* to be a critical point of average costs [differentiate $c(q)/q$ and see Exercise 5.D.1]. In the case where average cost $c(q)/q$ is U-shaped (i.e., with no critical point other than the global minimum, as shown in Figure 10.F.4), this implies that $q^* = \bar{q}$, and so $p^* = \bar{c}$ and $J^* = x(\bar{c})/\bar{q}$. Note, however, that the argument in the text does not require this assumption about the shape of average costs.

not intersect.[29] The nonexistence of competitive equilibrium can occur here for the same reason that we have already alluded to in small type in Section 10.C: The long-run production technologies we are considering exhibit nonconvexities.

It seems plausible, however, that when the efficient scale of a firm is small relative to the size of the market, this "integer problem" should not be too much of a concern. In fact, when we study oligopolistic markets in Chapter 12, we shall see that when firms' efficient scales are small in this sense, the oligopolistic equilibrium price is close to $\bar{c}$, the equilibrium price we would derive if we simply ignored the integer constraint on the number of firms J^*. Intuitively, when the efficient scale is small, we will have many firms in the industry and the equilibrium, although not strictly competitive, will involve a price close to $\bar{c}$. Thus, if the efficient scale is small relative to the size of the market [as measured by $x(\bar{c})$], then ignoring the integer problem and treating firms as price takers gives approximately the correct answer.

Third, when an equilibrium exists, as in Figure 10.F.4, the equilibrium outcome maximizes Marshallian aggregate surplus and therefore is Pareto optimal. To see this, note from Figure 10.F.4 that aggregate surplus at the considered equilibrium is equal to

$$\underset{x \geq 0}{\text{Max}} \int_0^x P(s)\, ds - \bar{c}x,$$

the maximized value of aggregate surplus when firms' cost functions are $\bar{c}q$. But because $c(q) \geq \bar{c}q$ for all q, this must be the largest attainable value of aggregate surplus given the actual cost function $c(\cdot)$; that is,

$$\underset{x > 0}{\text{Max}} \int_0^x P(s)\, ds - \bar{c}x \geq \int_0^{\hat{x}} P(s)\, ds - Jc(\hat{x}/J),$$

for all $\hat{x}$ and J. This fact provides an example of a point we raised at the end of Section 10.D (and will substantiate with considerable generality in Chapter 16): The first welfare theorem continues to be valid even in the absence of convexity of individual production sets.

Short-Run and Long-Run Comparative Statics

Although firms may enter and exit the market in response to profit opportunities in the long run, these changes may take time. For example, factories may need to be shut down, the workforce reduced, and machinery sold when a firm exits an industry. It may even pay a firm to continue operating until a suitable buyer for its plant and equipment can be found. When examining the comparative statics effects of a shock to a market, it is therefore important to distinguish between long-run and short-run effects.

Suppose, for example, that we are at a long-run equilibrium with J^* active firms

29. An intermediate case between constant returns (where any scale is efficient) and the case of a unique efficient scale occurs when there is a range $[\underline{q}, \bar{q}]$ of efficient scales (the average cost curve has a flat bottom). In this case, the integer problem is mitigated. For a long-run competitive equilibrium to exist, we now only need there to be some $q \in [\underline{q}, \bar{q}]$ such that $x(\bar{c})/q$ is an integer. Of course, as the interval $[\underline{q}, \bar{q}]$ grows larger, not only are the chances of a long-run equilibrium existing greater, but so are the chances of indeterminacy of the equilibrium number of firms (i.e., of multiple equilibria involving differing numbers of firms).

each producing q^* units of output and that there is some shock to demand (similar points can be made for supply shocks). In the short run, it may be impossible for any new firms to organize and enter the industry, and so we will continue to have J^* firms for at least some period of time. Moreover, these J^* firms may face a short-run cost function $c_s(\cdot)$ that differs from the long-run cost function $c(\cdot)$ because various input levels may be fixed in the short run. For example, firms may have the long-run cost function

$$c(q) = \begin{cases} K + \psi(q) & \text{if } q > 0 \\ 0 & \text{if } q = 0, \end{cases} \tag{10.F.1}$$

where $\psi(0) = 0$, $\psi'(q) > 0$, and $\psi''(q) > 0$. But in the short run, it may be impossible for an active firm to recover its fixed costs if it exits and sets $q = 0$. Hence, in the short run the firm has the cost function

$$c_s(q) = K + \psi(q) \qquad \text{for all } q \geq 0. \tag{10.F.2}$$

Another possibility is that $c(q)$ might be the cost function of some multiple-input production process, and in the short run an active firm may be unable to vary its level of some inputs. (See the discussion in Section 5.B on this point and also Exercises 10.F.5 and 10.F.6 for illustrations.)

Whenever the distinction between short run and long run is significant, the *short-run comparative statics effects* of a demand shock may best be determined by solving for the competitive equilibrium given J^* firms, each with cost function $c_s(\cdot)$, and the new demand function. This is just the equilibrium notion studied in Section 10.C, where we take firms' cost functions to be $c_s(\cdot)$. The *long-run comparative statics effects* can then be determined by solving for the long-run (i.e., free entry) equilibrium given the new demand function and long-run cost function $c(\cdot)$.

Example 10.F.1: *Short-Run and Long-Run Comparative Statics with Lumpy Fixed Costs that Are Sunk in the Short Run.* Suppose that the long-run cost function $c(\cdot)$ is given by (10.F.1) but that in the short run the fixed cost K is sunk so that $c_s(\cdot)$ is given by (10.F.2). The aggregate demand function is initially $x(\cdot, \alpha_0)$, and the industry is at a long-run equilibrium with J_0 firms, each producing $\bar{q}$ units of output [the efficient scale for cost function $c(\cdot)$], and a price of $p^* = \bar{c} = c(\bar{q})/\bar{q}$. This equilibrium position is depicted in Figure 10.F.5.

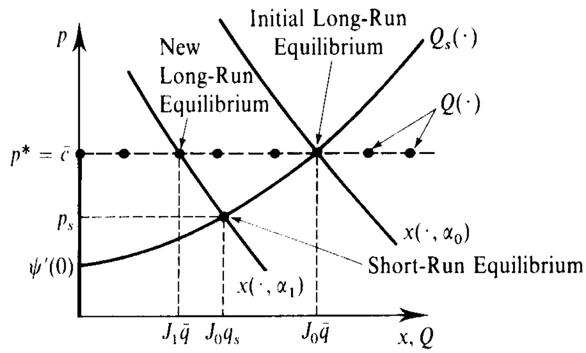


Figure 10.F.5
Short-run and long-run comparative statics in Example 10.F.1.

Now suppose that we have a shift to the demand function $x(\cdot, \alpha_1)$ shown in Figure 10.F.5. The short-run equilibrium is determined by the intersection of the graph of this demand function with the graph of the industry supply correspondence of the J_0 firms, each of which has short-run cost function $c_s(\cdot)$. The short-run aggregate supply correspondence is depicted as $Q_s(\cdot)$ in the figure. Thus, in the short run, the shock to demand causes price to fall to p_s and output per firm to fall to q_s. Firms' profits also fall; since $p_s < \bar{c}$, active firms lose money in the short run.

In the long run, however, firms exit in response to the decrease in demand, with the number of firms falling to $J_1 < J_0$, each producing output $\bar{q}$. Price returns to $p^* = \bar{c}$, aggregate consumption is $x(\bar{c}, \alpha_1)$, and all active firms once again earn zero profits. This new long-run equilibrium is also shown in Figure 10.F.5. ■

This division of dynamic adjustment into two periods, although useful as a first approximation, is admittedly crude. It may often be reasonable to think that there are several distinct short-run stages corresponding to different levels of adjustment costs associated with different decisions: in the very short run, production may be completely fixed; in the medium run, some inputs may be adjusted while others may not be; perhaps entry and exit take place only in the "very long run." Moreover, the methodology that we have discussed treats the two periods in isolation from each other. This approach ignores, for example, the possibility of intertemporal substitution by consumers when tomorow's price is expected to differ from today's (intertemporal substitution might be particularly important for very short-run periods when the fact that many production decisions are fixed can make prices very sensitive to demand shocks).

These weaknesses are not flaws in the competitive model per se, but rather only in the somewhat extreme methodological simplification adopted here. A fully satisfactory treatment of these issues requires an explicitly dynamic model that places expectations at center stage. In Chapter 20 we study dynamic models of competitive markets in greater depth. Nevertheless, this simple dichotomization into long-run and short-run periods of adjustment is often a useful starting point for analysis.

10.G Concluding Remarks on Partial Equilibrium Analysis

In principle, the analysis of Pareto optimal outcomes and competitive equilibria requires the simultaneous consideration of the entire economy (a task we undertake in Part IV). Partial equilibrium analysis can be thought of as facilitating matters on two accounts. On the positive side, it allows us to determine the equilibrium outcome in the particular market under study in isolation from all other markets. On the normative side, it allows us to use Marshallian aggregate surplus as a welfare measure that, in many cases of interest, has a very convenient representation in terms of the area lying vertically between the aggregate demand and supply curves.

In the model considered in Sections 10.C to 10.F, the validity of both of these simplifications rested, implicitly, on two premises: first, that the prices of all commodities other than the one under consideration remain fixed; second, that there are no wealth effects in the market under study. We devote this section to a few additional interpretative comments regarding these assumptions. (See also Section 15.E for an example illustrating the limits of partial equilibrium analysis.)

The assumption that the prices of goods other than the good under consideration (say, good ℓ) remain fixed is essential for limiting our positive and normative analysis to a single market. In Section 10.B, we justified this assumption in terms of the market for good ℓ being small and having a diffuse influence over the remaining markets. However, this is not its only possible justification. For example, the nonsubstitution theorem (see Appendix A of Chapter 5) implies that the prices of all other goods will remain fixed if the numeraire is the only primary (i.e., nonproduced) factor, all produced goods other than ℓ are produced under conditions of constant returns using the numeraire and produced commodities other than ℓ as inputs, and there is no joint production.[30]

Even when we cannot assume that all other prices are fixed, however, a generalization of our single-market partial equilibrium analysis is sometimes possible. Often we are interested not in a single market but in a group of commodities that are strongly interrelated either in consumers' tastes (tea and coffee are the classic examples) or in firms' technologies. In this case, studying one market at a time while keeping other prices fixed is no longer a useful approach because what matters is the *simultaneous* determination of *all* prices in the group. However, if the prices of goods outside the group may be regarded as unaffected by changes within the markets for this group of commodities, and if there are no wealth effects for commodities in the group, then we can extend much of the analysis presented in Sections 10.C to 10.F.

To this effect, suppose that the group is composed of M goods, and let $x_i \in \mathbb{R}^M_+$ and $q_j \in \mathbb{R}^M$ be vectors of consumptions and productions for these M goods. Each consumer has a utility function of the form

$$u_i(m_i, x_i) = m_i + \phi_i(x_i),$$

where m_i is the consumption of the numeraire commodity (i.e., the total expenditure on commodities outside the group). Firms' cost functions are $c_j(q_j)$. With this specification, many of the basic results of the previous sections go through unmodified (often it is just a matter of reinterpreting x_i and q_j as vectors). In particular, the results discussed in Section 10.C on the uniqueness of equilibrium and its independence from initial endowments still hold (see Exercise 10.G.1), as do the welfare theorems of Section 10.D. However, our ability to conduct welfare analysis using the areas lying vertically between demand and supply curves becomes much more limited. The cross-effects among markets with changing and interrelated prices cannot be

30. A simple example of this result arises when all produced goods other than ℓ are produced directly from the numeraire with constant returns to scale. In this case, the equilibrium price of each of these goods is equal to the amount of the numeraire that must be used as an input in its production per unit of output produced. More generally, prices for produced goods other than ℓ will remain fixed under the conditions of the nonsubstitution theorem because all efficient production vectors can be generated using a single set of techniques. In any equilibrium, the price of each produced good other than ℓ must be equal to the amount of the numeraire embodied in a unit of the good in the efficient production technique, either directly through the use of the numeraire as an input or indirectly through the use as inputs of produced goods other than ℓ that are in turn produced using the numeraire (or using other produced goods that are themselves produced using the numeraire, and so on).

ignored.[31] (Exercises 10.G.3 to 10.G.5 ask you to consider some issues related to this point.)

The assumption of no wealth effects for good ℓ, on the other hand, is critical for the validity of the style of welfare analysis that we have carried out in this chapter. Without it, as we shall see in Part IV, Pareto optimality cannot be determined independently from the particular distribution of welfare sought, and we already know from Section 3.I that area measures calculated from Walrasian demand functions are not generally correct measures of compensating or equivalent variations (for which the Hicksian demand functions should be used). However, the assumption of no wealth effects is much less critical for positive analysis (determination of equilibrium, comparative statics effects, and so on). Even with wealth effects, the demand-and-supply apparatus can still be quite helpful for the positive part of the theory. The behavior of firms, for example, is not changed in any way. Consumers, on the other hand, have a demand function that, with prices of the other goods kept fixed, now depends only on the price for good ℓ and wealth. If wealth is determined from initial endowments and shareholdings, then we can view wealth as itself a function of the price of good ℓ (recall that other prices are fixed), and so we can again express demand as a function of this good's price alone. Formally, the analysis reduces to that presented in Section 10.C: The equilibrium in market ℓ can be identified as an intersection point of demand and supply curves.[32]

31. A case in which the single-market analysis for good ℓ is still fully justified is when utility and cost functions have the form

$$u_i(m_i, x_i) = m_i + \phi_{\ell i}(x_{\ell i}) + \phi_{-\ell,i}(x_{-\ell,i}),$$

and

$$c_j(q_j) = c_{\ell j}(q_{\ell j}) + c_{-\ell,j}(q_{-\ell,j}),$$

where $x_{-\ell,i}$ and $q_{-\ell,j}$ are consumption and production vectors for goods in the group other than ℓ. With this additive separability in good ℓ, the markets for goods in the group other than ℓ do not influence the equilibrium price in market ℓ. Good ℓ is effectively independent of the group, and we can treat it in isolation, as we have done in the previous sections. (In point of fact, we do not even need to assume that the remaining markets in the group keep their prices fixed. What happens in them is simply irrelevant for equilibrium and welfare analysis in the market for good ℓ.) See Exercise 10.G.2.

32. The presence of wealth effects can lead, however, to some interesting new phenomena on the consumer's side. One is the *backward-bending* demand curve, where demand for a good is *increasing* in its price over some range. This can happen if consumers have endowments of good ℓ, because then an increase in its price increases consumers' wealth and could lead to a net increase in their demands for good ℓ, even if it is a normal good.

REFERENCES

Marshall, A. (1920). *Principles of Economics*. New York: Macmillan.

Stigler, G. (1987). *The Theory of Price*, 4th ed. New York: Macmillan.

Vives, X. (1987). Small income effects: A Marshallian theory of consumer surplus and downward sloping demand. *Review of Economic Studies* **54**: 87–103.

EXERCISES

10.B.1[B] The concept defined in Definition 10.B.2 is sometimes known as *strong Pareto efficiency*. An outcome is *weakly Pareto efficient* if there is no alternative feasible allocation that makes *all* individuals *strictly* better off.

(a) Argue that if an outcome is strongly Pareto efficient, then it is weakly Pareto efficient as well.

(b) Show that if all consumers' preferences are continuous and strongly monotone, then these two notions of Pareto efficiency are equivalent for any *interior* outcome (i.e., an outcome in which each consumer's consumption lies in the interior of his consumption set). Assume for simplicity that $X_i = \mathbb{R}^L_+$ for all i.

(c) Construct an example where the two notions are not equivalent. Why is the strong monotonicity assumption important in **(b)**? What about interiority?

10.B.2[A] Show that if allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and price vector $p^* \gg 0$ constitute a competitive equilibrium, then allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and price vector αp^* also constitute a competitive equilibrium for any scalar $\alpha > 0$.

10.C.1[B] Suppose that consumer i's preferences can be represented by the utility function $u_i(x_{1i}, \dots, x_{Li}) = \sum_\ell \log(x_{\ell i})$ (these are Cobb–Douglas preferences).

(a) Derive his demand for good ℓ. What is the wealth effect?

(b) Now consider a sequence of situations in which we proportionately increase both the number of goods and the consumer's wealth. What happens to the wealth effect in the limit?

10.C.2[B] Consider the two-good quasilinear model presented in Section 10.C with one consumer and one firm (so that $I = 1$ and $J = 1$). The initial endowment of the numeraire is $\omega_m > 0$, and the initial endowment of good ℓ is 0. Let the consumer's quasilinear utility function be $\phi(x) + m$, where $\phi(x) = \alpha + \beta \ln x$ for some $(\alpha, \beta) \gg 0$. Also, let the firm's cost function be $c(q) = \sigma q$ for some scalar $\sigma > 0$. Assume that the consumer receives all the profits of the firm. Both the firm and the consumer act as price takers. Normalize the price of good m to equal 1, and denote the price of good ℓ by p.

(a) Derive the consumer's and the firm's first-order conditions.

(b) Derive the competitive equilibrium price and output of good ℓ. How do these vary with α, β, and σ?

10.C.3[B] Consider a central authority who operates J firms with differentiable convex cost functions $c_j(q_j)$ for producing good ℓ from the numeraire. Define $C(q)$ to be the central authority's minimized cost level for producing aggregate quantity q; that is

$$C(q) = \underset{(q_1, \dots, q_J) \geq 0}{\text{Min}} \quad \sum_{j=1}^{J} c_j(q_j)$$
$$\text{s.t.} \quad \sum_{j=1}^{J} q_j \geq q.$$

(a) Derive the first-order conditions for this cost-minimization problem.

(b) Show that at the cost-minimizing production allocation $(q_1^*, \dots, q_J^*)$, $C'(q) = c_j'(q_j^*)$ for all j with $q_j^* > 0$ (i.e., the central authority's marginal cost at aggregate output level q equals each firm's marginal cost level at the optimal production allocation for producing q).

(c) Show that if firms all maximize profit facing output price $p = C'(q)$ (with the price of the numeraire equal to 1), then the consequent output choices result in an aggregate output of q. Conclude that $C'(\cdot)$ is the inverse of the industry supply function $q(\cdot)$.

10.C.4[B] Consider a central authority who has x units of good ℓ to allocate among I consumers, each of whom has a quasilinear utility function of the form $\phi_i(x_i) + m_i$, with $\phi_i(\cdot)$ a differentiable, increasing, and strictly concave function. The central authority allocates good ℓ to maximize the sum of consumers' utilities $\sum_i u_i$.

(a) Set up the central authority's problem and derive its first-order condition.

(b) Let $\gamma(x)$ be the value function of the central authority's problem, and let $P(x) = \gamma'(x)$ be its derivative. Show that if $(x_1^*, \ldots, x_I^*)$ is the optimal allocation of good ℓ given available quantity x, then $P(x) = \phi_i'(x_i^*)$ for all i with $x_i^* > 0$.

(c) Argue that if all consumers maximize utility facing a price for good ℓ of $P(x)$ (with the price of the numeraire equal to 1), then the aggregate demand for good ℓ is exactly x. Conclude that $P(\cdot)$ is, in fact, the inverse of the aggregate demand function $x(\cdot)$.

10.C.5[B] Derive the differential change in the equilibrium price in response to a differential change in the tax in Example 10.C.1 by applying the implicit function theorem to the system of equations (10.C.4) to (10.C.6).

10.C.6[B] A tax is to be levied on a commodity bought and sold in a competitive market. Two possible forms of tax may be used: In one case, a *specific* tax is levied, where an amount t is paid per unit bought or sold (this is the case considered in the text); in the other case, an *ad valorem* tax is levied, where the government collects a tax equal to τ times the amount the seller receives from the buyer. Assume that a partial equilibrium approach is valid.

(a) Show that, with a specific tax, the ultimate cost of the good to consumers and the amounts purchased are independent of whether the consumers or the producers pay the tax.

(b) Show that this is not generally true with an ad valorem tax. In this case, which collection method leads to a higher cost to consumers? Are there special cases in which the collection method is irrelevant with an ad valorem tax?

10.C.7[B] An ad valorem tax of τ (see Exercise 10.C.6 for a definition) is to be levied on consumers in a competitive market with aggregate demand curve $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve $q(p) = \alpha p^{\gamma}$, where $\alpha > 0$ and $\gamma > 0$. Calculate the percentage change in consumer cost and producer receipts per unit sold for a small ("marginal") tax. Denote $\kappa = (1 + \tau)$. Assume that a partial equilibrium approach is valid.

Compute the elasticity of the equilibrium price with respect to κ. Argue that when $\gamma = 0$ producers bear the full effect of the tax while consumers' total costs of purchase are unaffected, and that when $\varepsilon = 0$ it is consumers who bear the full burden of the tax. What happens when each of these elasticities approaches ∞ in absolute value?

10.C.8[B] Suppose that there are J firms producing good ℓ, each with a differentiable cost function $c(q, \alpha)$ that is strictly convex in q, where α is an exogenous parameter that affects costs (it could be a technological parameter or an input price). Assume that $\partial c(q, \alpha)/\partial \alpha > 0$. The differentiable aggregate demand function for good ℓ is $x(p)$, with $x'(\cdot) \leq 0$. Assume that partial equilibrium analysis is justified.

Let $q^*(\alpha)$ be the *per firm* output and $p^*(\alpha)$ be the equilibrium price in the competitive equilibrium given α.

(a) Derive the marginal change in a firm's profits with respect to α.

(b) Give the weakest possible sufficient condition, stated in terms of marginal and average costs and/or their derivatives, that guarantees that if α increases marginally, then firms' equilibrium profits decline for any demand function $x(\cdot)$ having $x'(\cdot) \leq 0$. Show that if this condition is not satisfied, then there are demand functions such that profits increase when α increases.

(c) In the case where α is the price of factor input k, interpret the condition in **(b)** in terms of the conditional factor demand for input k.

10.C.9^B Suppose that in a partial equilibrium context there are J identical firms that produce good ℓ with cost function $c(w, q)$, where w is a vector of factor input prices. Show that an increase in the price of factor k, w_k, lowers the equilibrium price of good ℓ if and only if factor k is an *inferior* factor, that is, if at fixed input prices, the use of factor k is decreasing in a firm's output level.

10.C.10^B Consider a market with demand curve $x(p) = \alpha p^{\varepsilon}$ and with J firms, each of which has marginal cost function $c'(q) = \beta q^{\eta}$, where $(\alpha, \beta, \eta) \gg 0$ and $\varepsilon < 0$. Calculate the competitive equilibrium price and output levels. Examine the comparative statics change in these variables as a result of changes in α and β. How are these changes affected by ε and η?

10.C.11^B Assume that partial equilibrium analysis is valid. Suppose that firms 1 and 2 are producing a positive level of output in a competitive equilibrium. The cost function for firm j is given by $c(q, \alpha_j)$, where α_j is an exogenous technological parameter. If α_1 differs from α_2 marginally, what is the difference in the two firms' profits?

10.D.1^B Prove that under the assumptions that the $\phi_i(\cdot)$ functions are strictly concave and the cost functions $c_j(\cdot)$ are convex, the optimal individual consumption levels of good ℓ in problem (10.D.2) are uniquely defined. Conclude that the optimal aggregate production level of good ℓ is therefore also uniquely defined. Show that if the cost functions $c_j(\cdot)$ are *strictly* convex, then the optimal individual production levels of good ℓ in problem (10.D.2) are also uniquely defined.

10.D.2^B Determine the optimal consumption and production levels of good ℓ for the economy described in Exercise 10.C.2. Compare these with the equilibrium levels you identified in that exercise.

10.D.3^B In the context of the two-good quasilinear economy studied in Section 10.D, show that any allocation that is a solution to problem (10.D.6) is Pareto optimal and that any Pareto optimal allocation is a solution to problem (10.D.6) for *some* choice of utility levels $(\bar{u}_2, \ldots, \bar{u}_I)$.

10.D.4^B Derive the first-order conditions for problem (10.D.6) and compare them with conditions (10.D.3) to (10.D.5).

10.E.1^C Suppose that $J_d > 0$ of the firms that produce good ℓ are domestic firms, and $J_f > 0$ are foreign firms. All domestic firms have the same convex cost function for producing good ℓ, $c_d(q_j)$. All foreign firms have the same convex cost function $c_f(q_j)$. Assume that partial equilibrium analysis is valid.

The government of the domestic country is considering imposing a per-unit tariff of τ on imports of good ℓ. The government wants to maximize domestic welfare as measured by the *domestic* Marshallian surplus (i.e., the sum of domestic consumers' utilities less domestic firms' costs).

(a) Show that if $c_f(\cdot)$ is strictly convex, then imposition of a small tariff raises domestic welfare.

(b) Show that if $c_f(\cdot)$ exhibits constant returns to scale, then imposition of a small tariff lowers domestic welfare.

10.E.2^B Consumer surplus when consumers face effective price $\hat{p}$ can be written as

$$CS(\hat{p}) = \int_0^{x(\hat{p})} [P(s) - \hat{p}]\, ds.$$

Prove by means of a change of variables and integration by parts that this integral is equal to $\int_{\bar{p}}^{\infty} x(s)\,ds$.

10.E.3[C] (*Ramsey tax problem*) Consider a fully separable quasilinear model with L goods in which each consumer has preferences of the form $u_i(x_i) = x_{1i} + \sum_{\ell=2}^{L} \phi_{\ell i}(x_{\ell i})$ and each good $2, \ldots, L$ is produced with constant returns to scale from good 1, using c_ℓ units of good 1 per unit of good ℓ produced. Assume that consumers initially hold endowments only of the numeraire, good 1. Hence, consumers are net sellers of good 1 to the firms and net purchasers of goods $2, \ldots, L$.

In this setting, consumer i's demand for each good $\ell \neq 1$ can be written in the form $x_{\ell i}(p_\ell)$, so that demand for good ℓ is independent of the consumer's wealth and all other prices, and welfare can be measured by the sum of the Marshallian aggregate surpluses in the $L - 1$ markets for nonnumeraire commodities (see Section 10.G and Exercise 10.G.2 for more on this).

Suppose that the government must raise R units of good 1 through (specific) commodity taxes. Note, in particular, that such taxes involve taxing a *transaction* of a good, *not* an individual's consumption level of that good.

Let t_ℓ denote the tax to be paid by a consumer in units of good 1 for each unit of good $\ell \neq 1$ purchased, and let t_1 be the tax in units of good 1 to be paid by consumers for each unit of good 1 sold to a firm. Normalize the price paid by firms for good 1 to equal 1. Under our assumptions, each choice of $t = (t_1, \ldots, t_L)$ results in a consumer paying a total of $c_\ell + t_\ell$ per unit of good $\ell \neq 1$ purchased and having to part with $(1 + t_1)$ units of good 1 for each unit of good 1 sold to a firm.

(a) Consider two possible tax vectors t and t'. Show that if t' is such that $(c_\ell + t'_\ell) = \alpha(c_\ell + t_\ell)$ and $(1 + t'_1) = (1/\alpha)(1 + t_1)$ for some scalar $\alpha > 0$, then the two sets of taxes raise the same revenue. Conclude from this fact that the government can restrict attention to tax vectors that leave one good untaxed.

(b) Let good 1 be the untaxed good (i.e., set $t_1 = 0$). Derive conditions describing the taxes that should be set on goods $2, \ldots, L$ if the government wishes to minimize the welfare loss arising from this taxation. Express this formula in terms of the elasticity of demand for each good.

(c) Under what circumstances should the tax rate on all goods be equal? In general, which goods should have higher tax rates? When would taxing only good 1 be optimal?

10.F.1[A] Show that if $c(q)$ is strictly convex in q and $c(0) = 0$, then $\pi(p) > 0$ if and only if $p > c'(0)$.

10.F.2[B] Consider a market with demand function $x(p) = A - Bp$ in which every potential firm has cost function $c(q) = K + \alpha q + \beta q^2$, where $\alpha > 0$ and $\beta > 0$.

(a) Calculate the long-run competitive equilibrium price, output per firm, aggregate output, and number of firms. Ignore the integer constraint on the number of firms. How does each of these vary with A?

(b) Now examine the short-run competitive equilibrium response to a change in A starting from the long-run equilibrium you identified in (a). How does the change in price depend on the level of A in the initial equilibrium? What happens as $A \to \infty$? What accounts for this effect of market size?

10.F.3[B] (D. Pearce) Consider a partial equilibrium setting in which each (potential) firm has a long-run cost function $c(\cdot)$, where $c(q) = K + \phi(q)$ for $q > 0$ and $c(0) = 0$. Assume that $\phi'(q) > 0$ and $\phi''(q) < 0$, and denote the firm's efficient scale by $\bar{q}$. Suppose that there is initially a long-run equilibrium with J^* firms. The government considers imposing two different types of taxes: The first is an ad valorem tax of τ (see Exercise 10.C.6) on sales of the good. The second is a tax T that must be paid by any operating firm (where a firm is considered to be "operating" if it sells a positive amount). If the two taxes would raise an equal amount of revenue with the initial level of sales and number of firms, which will raise more after the industry adjusts to a new long-run equilibrium? (You should ignore the integer constraint on the number of firms.)

10.F.4$^{\mathbf{B}}$ (J. Panzar) Assume that partial equilibrium analysis is valid. The single-output, many-input technology for producing good ℓ has a differentiable cost function $c(w, q)$, where $w = (w_1, \ldots, w_K)$ is a vector of factor input prices and q is the firm's output of good ℓ. Given factor prices w, let $\bar{q}(w)$ denote the firm's efficient scale. Assume that $\bar{q}(w) > 0$ for all w. Also let $p_\ell^*(w)$ denote the long-run equilibrium price of good ℓ when factor prices are w. Show that the function $p_\ell^*(w)$ is nondecreasing, homogeneous of degree one, and concave. (You should ignore the integer constraint on the number of firms.)

10.F.5$^{\mathbf{C}}$ Suppose that there are J firms that can produce good ℓ from K factor inputs with differentiable cost function $c(w, q)$. Assume that this function is strictly convex in q. The differentiable aggregate demand function for good ℓ is $x(p, \alpha)$, where $\partial x(p, \alpha)/\partial p < 0$ and $\partial x(p, \alpha)/\partial \alpha > 0$ (α is an exogenous parameter affecting demand). However, although $c(w, q)$ is the cost function when all factors can be freely adjusted, factor k cannot be adjusted in the short run.

Suppose that we are initially at an equilibrium in which all inputs are optimally adjusted to the equilibrium level of output q^* and factor prices w so that, letting $z_k(w, q)$ denote a firm's conditional factor demand for input k when all inputs can be adjusted, $z_k^* = z_k(w, q^*)$.

(a) Show that a firm's equilibrium response to an increase in the price of good ℓ is larger in the long run than in the short run.

(b) Show that this implies that the long-run equilibrium response of p_ℓ to a marginal increase in α is smaller than the short-run response. Show that the reverse is true for the response of the equilibrium aggregate consumption of good ℓ (hold the number of firms equal to J in both the short run and long run).

10.F.6$^{\mathbf{B}}$ Suppose that the technology for producing a good uses capital (z_1) and labor (z_2) and takes the Cobb–Douglas form $f(z_1, z_2) = z_1^\alpha z_2^{1-\alpha}$, where $\alpha \in (0, 1)$. In the long run, both factors can be adjusted; but in the short run, the use of capital is fixed. The industry demand function takes the form $x(p) = a - bp$. The vector of input prices is (w_1, w_2). Find the long-run equilibrium price and aggregate quantity. Holding the number of firms and the level of capital fixed at their long-run equilibrium levels, what is the short-run industry supply function?

10.F.7$^{\mathbf{B}}$ Consider a case where in the short run active firms can increase their use of a factor but cannot decrease it. Show that the short-run cost curve will exhibit a kink (i.e., be nondifferentiable) at the current (long-run) equilibrium. Analyze the implications of this fact for the relative variability of short-run prices and quantities.

10.G.1$^{\mathbf{B}}$ Consider the case of an interrelated group of M commodities. Let consumer i's utility function take the form $u_i(x_{1i}, \ldots, x_{Mi}) = m_i + \phi_i(x_{1i}, \ldots, x_{Mi})$. Assume that $\phi_i(\cdot)$ is differentiable and strictly concave. Let firm j's cost function be the differentiable convex function $c_j(q_{1j}, \ldots, q_{Mj})$.

Normalize the price of the numeraire to be 1. Derive $(I + J + 1)M$ equations characterizing the $(I + J + 1)M$ equilibrium quantities $(x_{1i}^*, \ldots, x_{Mi}^*)$ for $i = 1, \ldots, I$, $(q_{1j}^*, \ldots, q_{Mj}^*)$ for $j = 1, \ldots, J$, and $(p_1^*, \ldots, p_M^*)$. [*Hint*: Derive consumers' and firms' first-order conditions and the $M - 1$ market-clearing conditions in parallel to our analysis of the single-market case.] Argue that the equilibrium prices and quantities of these M goods are independent of consumers' wealths, that equilibrium individual consumptions and aggregate production levels are unique, and that if the $c_j(\cdot)$ functions are strictly convex, then equilibrium individual production levels are also unique.

10.G.2^B Consider the case in which the functions $\phi_i(\cdot)$ and $c_j(\cdot)$ in Exercise 10.G.1 are separable in good ℓ (one of the goods in the group): $\phi_i(\cdot) = \phi_{\ell i}(x_{\ell i}) + \phi_{-\ell,i}(x_{-\ell,i})$ and $c_j(\cdot) = c_{\ell j}(q_{\ell j}) + c_{-\ell,j}(q_{-\ell,j})$. Argue that in this case, the equilibrium price, consumption, and production of good ℓ can be determined independently of other goods in the group. Also argue that under the same assumptions as in the single-market case studied in Section 10.E, changes in welfare caused by changes in the market for this good can be captured by the Marshallian aggregate surplus for this good, $\sum_i \phi_{\ell i}(x_{\ell i}) - \sum_j c_{\ell j}(q_{\ell j})$, which can be represented in terms of the areas lying vertically between the demand and supply curves for good ℓ. Note the implication of these results for the case in which we have separability of all goods: $\phi_i(\cdot) = \sum_\ell \phi_{\ell i}(x_{\ell i})$ and $c_j(\cdot) = \sum_\ell c_{\ell j}(q_{\ell j})$.

10.G.3^B Consider a three-good economy ($\ell = 1, 2, 3$) in which every consumer has preferences that can be described by the utility function $u(x) = x_1 + \phi(x_2, x_3)$ and there is a single production process that produces goods 2 and 3 from good 1 having $c(q_2, q_3) = c_2 q_2 + c_3 q_3$. Suppose that we are considering a tax change in only a single market, say market 2.

(a) Show that if the price in market 3 is undistorted (i.e., if $t_3 = 0$), then the change in aggregate surplus caused by the tax change can be captured solely through the change in the area lying vertically between market 2's demand and supply curves holding the price of good 3 at its initial level.

(b) Show that if market 3 is initially distorted because $t_3 > 0$, then by using only the single-market measure in **(a)**, we would overstate the decrease in aggregate surplus if good 3 is a substitute for good 2 and would understate it if good 3 is a complement. Provide an intuitive explanation of this result. What is the correct measure of welfare change?

10.G.4^B Consider a three-good economy ($\ell = 1, 2, 3$) in which every consumer has preferences that can be described by the utility function $u(x) = x_1 + \phi(x_2, x_3)$ and there is a single production process that produces goods 2 and 3 from good 1 having $c(q_2, q_3) = c_2 q_2 + c_3 q_3$. Derive an expression for the welfare loss from an increase in the tax rates on both goods.

10.G.5^B Consider a three-good economy ($\ell = 1, 2, 3$) in which every consumer has preferences that can be described by the utility function $u(x) = x_1 + \phi(x_2, x_3)$ and there is a single production process that produces goods 2 and 3 from good 1 having $c(q_2, q_3) = c_2(q_2) + c_3(q_3)$, where $c_2(\cdot)$ and $c_3(\cdot)$ are strictly increasing and strictly convex.

(a) If goods 2 and 3 are substitutes, what effect does an increase in the tax on good 2 have on the price paid by consumers for good 3? What if they are complements?

(b) What is the bias from applying the formula for welfare loss you derived in part (b) of Exercise 10.G.3 using the price paid by consumers for good 3 prior to the tax change in both the case of substitutes and that of complements?