PART FOUR

General Equilibrium

Part IV is devoted to an examination of competitive market economies from a *general equilibrium* perspective. Our use of the term "general equilibrium" refers both to a methodological point of view and to a substantive theory.

Methodologically, the general equilibrium approach has two central features. First, it views the economy as a *closed* and *interrelated* system in which we must simultaneously determine the equilibrium values of all variables of interest. Thus, when we evaluate the effects of a perturbation in the economic environment, the equilibrium levels of the entire set of endogenous variables in the economy needs to be recomputed. This stands in contrast to the *partial equilibrium* approach, where the impact on endogenous variables not directly related to the problem at hand is explicitly or implicitly disregarded.

A second central feature of the general equilibrium approach is that it aims at reducing the set of variables taken as exogenous to a small number of physical realities (e.g., the set of economic agents, the available technologies, the preferences and physical endowments of goods of various agents).

From a substantive viewpoint, general equilibrium theory has a more specific meaning: It is a theory of the determination of equilibrium prices and quantities in a system of perfectly competitive markets. This theory is often referred to as the *Walrasian theory* of markets [from L. Walras (1874)], and it is the object of our study in Part IV. The Walrasian theory of markets is very ambitious. It attempts no less than to predict the complete vector of final consumptions and productions using only the fundamentals of the economy (the list of commodities, the state of technology, preferences and endowments), the institutional assumption that a price is quoted for every commodity (including those that will not be traded at equilibrium), and the behavioral assumption of price taking by consumers and firms.

Strictly speaking, we introduced a particular case of the general equilibrium model in Chapter 10. There, we carried out an equilibrium and welfare analysis of perfectly competitive markets under the assumption that consumers had quasilinear preferences. In that setting, consumer demand functions do not display wealth effects (except for a single commodity, called the *numeraire*); as a consequence, the analysis of a single market (or small group of markets) could be pursued in a manner understandable as traditional partial equilibrium analysis. A good deal of what we do in Part IV can be viewed as an attempt to extend the ideas of Chapter 10 to a world in which wealth effects are significant. The primary motivation for this is the increase in realism it brings. To make practical use of equilibrium analysis for studying the performance of an entire economy, or for evaluating policy interventions that affect large numbers of markets simultaneously, wealth effects, a primary source of linkages across markets, cannot be neglected, and therefore the general equilibrium approach is essential.

Although knowledge of the material discussed in Chapter 10 is not a strict prerequisite for Part IV, we nonetheless strongly recommend that you study it, especially Sections 10.B to 10.D. It constitutes an introduction to the main issues and provides a simple and analytically very useful example. We will see in the different chapters of Part IV that quite a number of the important results established in Chapter 10 for the quasilinear situation carry over to the case of general preferences. But many others do not. To understand why this may be so, recall from Chapters 4 and 10 that a group of consumers with quasilinear preferences (with respect to the same numeraire) admits the existence of a (normative) representative consumer. This is a powerful restriction on the behavior of aggregate demand that will not be available to us in the more general settings that we study here.

It is important to note that, relative to the analysis carried out in Part III, we incur a cost for accomplishing the task that general equilibrium sets itself to do: the assumptions of price-taking behavior and universal price quoting—that is, the existence of markets for every relevant commodity (with the implication of symmetric information)—are present in nearly all the theory studied in Part IV. Thus, in many respects, we are not going as deep as we did in Part III in the microanalysis of markets, of market failure, and of the strategic interdependence of market actors. The trade-off in conceptual structure between Parts III and IV reflects, in a sense, the current state of the frontier of microeconomic research.

The content of Part IV is organized into six chapters.

Chapter 15 presents a preliminary discussion. Its main purpose is to illustrate the issues that concern general equilibrium theory by means of three simple examples: the *two-consumer Edgeworth box economy*; the *one-consumer, one-firm economy*, and the *small open economy* model.

Chapters 16 and 17 constitute the heart of the formal analysis in Part IV. Chapter 16 presents the formal structure of the general equilibrium model and introduces two central concepts of the theory: the notions of *Pareto optimality* and *price-taking equilibrium* (and, in particular, *Walrasian equilibrium*). The chapter is devoted to the examination of the relationship between these two concepts. The emphasis is therefore *normative*, focusing on the welfare properties of price-taking equilibria. The core of the chapter is concerned with the formulation and proof of the two *fundamental theorems of welfare economics*.

In Chapter 17, the emphasis is, instead, on *positive* (or *descriptive*) properties of Walrasian equilibria. We study a number of questions pertaining to the predictive power of the Walrasian theory, including the existence, local and global uniqueness, and comparative statics behavior of Walrasian equilibria.

Chapters 18 to 20 explore extensions of the basic analysis presented in Chapters 16 and 17. Chapter 18 covers a number of topics whose origins lie in normative theory or the cooperative theory of games; these topics share the feature that they provide a deeper look at the foundations of price-taking equilibria by exploiting properties derived from the mass nature of markets. We study the important *core equivalence theorem*, examine further the idea of Walrasian equilibria as the limit of noncooperative equilibria as markets grow large (a subject already broached in Section 12.F), and present two normative characterizations of Walrasian equilibria: one in terms of *envy-freeness* (or *anonymity*) and the other in terms of a *marginal productivity principle*. Appendix A of Chapter 18 offers a brief introduction to the cooperative theory of games.

Chapter 19 covers the modeling of uncertainty in a general equilibrium context. The ability to do this in a theoretically satisfying way has been one of the success stories of general equilibrium theory. The concepts of *contingent commodities*, *Arrow–Debreu equilibrium*, *sequential trade* (in a two-period setting), *Radner equilibrium*, *arbitrage*, *rational expectations equilibrium*, and *incomplete markets* are all introduced and studied here. The chapter provides a natural link to the modern theory of finance.

Chapter 20 considers the application of the general theory to dynamic competitive economies (but with no uncertainty) and also studies a number of issues specific to this environment. Notions such as *impatience*, *dynamic efficiency*, and *myopic* versus *overall utility maximization* are introduced. The chapter first analyzes dynamic representative consumer economies (including the *Ramsey–Solow model*), then generalizes to the case of a finite number of consumers, and concludes with a brief presentation of the *overlapping generations model*. In the process, we explore a wide range of dynamic behaviors. The chapter provides a natural link to macroeconomic theory.

The modern classics of general equilibrium theory are Debreu (1959) and Arrow and Hahn (1971). These texts provide further discussion of topics treated here. For extensions, we recommend the encyclopedic coverage of Arrow and Intriligator (1981, 1982, 1986) and Hildenbrand and Sonnenschein (1991). See also the more recent textbook account of Ellickson (1993). General equilibrium analysis has a very important applied dimension that we do not touch on in this book but that accounts in good part for the importance of the theory. For a review, we recommend Shoven and Whalley (1992).

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CHAPTER

15

General Equilibrium Theory: Some Examples

15.A Introduction

The purpose of this chapter is to present a preliminary discussion. In it, we describe and analyze three simple examples of general equilibrium models. These examples introduce some of the questions, concepts, and common techniques that will occupy us for the rest of Part IV.

In most economies, three basic economic activities occur: production, exchange, and consumption. In Section 15.B, we restrict our focus to exchange and consumption. We analyze the case of a *pure exchange economy*, in which no production is possible and the commodities that are ultimately consumed are those that individuals possess as *endowments*. Individuals trade these endowments among themselves in the marketplace for mutual advantage. The model we present is the simplest-possible exchange problem: two consumers trading two goods between each other. In this connection, we introduce an extremely handy graphical device, the *Edgeworth box*.

In Section 15.C, we introduce production by studying an economy formed by one firm and one consumer. Using this simple model, we explore how the production and consumption sides of the economy fit together.

In Section 15.D, we examine the production side of the economy in greater detail by discussing the allocation of resources among several firms. To analyze this issue in isolation, we study the case of a small open economy that takes the world prices of its outputs as fixed, a central model in international trade literature.

Section 15.E illustrates, by means of an example, some of the potential dangers of adopting a partial equilibrium perspective when a general equilibrium approach is called for.

As we noted in the introduction of Part IV, Chapter 10 contains another simple example of a general equilibrium model: that of an economy in which consumers have preferences admitting a quasilinear representation.

15.B Pure Exchange: The Edgeworth Box

A *pure exchange economy* (or, simply, an *exchange economy*) is an economy in which there are no production opportunities. The economic agents of such an economy are consumers who possess initial stocks, or *endowments*, of commodities. Economic activity consists of trading and consumption.

The simplest economy with the possibility of profitable exchange is one with two commodities and two consumers. As it turns out, this case is amenable to analysis by a graphical device known as the *Edgeworth box*, which we use extensively in this section. Throughout, we assume that the two consumers act as price takers. Although this may not seem reasonable with only two traders, our aim here is to illustrate some of the features of general equilibrium models in the simplest-possible way.[1]

To begin, assume that there are two consumers, denoted by $i = 1, 2$, and two commodities, denoted by $\ell = 1, 2$. Consumer i's consumption vector is $x_i = (x_{1i}, x_{2i})$; that is, consumer i's consumption of commodity ℓ is $x_{\ell i}$. We assume that consumer i's consumption set is $\mathbb{R}^2_+$ and that he has a preference relation $\succsim_i$ over consumption vectors in this set. Each consumer i is initially endowed with an amount $\omega_{\ell i} \geq 0$ of good ℓ. Thus, consumer i's *endowment vector* is $\omega_i = (\omega_{1i}, \omega_{2i})$. The *total endowment* of good ℓ in the economy is denoted by $\bar{\omega}_\ell = \omega_{\ell 1} + \omega_{\ell 2}$; we assume that this quantity is strictly positive for both goods.

An *allocation* $x \in \mathbb{R}^4_+$ in this economy is an assignment of a nonnegative consumption vector to each consumer: $x = (x_1, x_2) = ((x_{11}, x_{21}), (x_{12}, x_{22}))$. We say that an allocation is *feasible* for the economy if

$$x_{\ell 1} + x_{\ell 2} \leq \bar{\omega}_\ell \qquad \text{for } \ell = 1, 2, \tag{15.B.1}$$

that is, if the total consumption of each commodity is no more than the economy's aggregate endowment of it (note that in this notion of feasibility, we are implicitly assuming that there is free disposal of commodities).

The feasible allocations for which equality holds in (15.B.1) could be called *nonwasteful*. Nonwasteful feasible allocations can be depicted by means of an *Edgeworth box*, shown in Figure 15.B.1. In the Edgeworth box, consumer 1's quantities are measured in the usual way, with the southwest corner as the origin. In contrast, consumer 2's quantities are measured using the northeast corner as the origin. For both consumers, the vertical dimension measures quantities of good 2, and the horizontal dimension measures quantities of good 1. The length of the box is $\bar{\omega}_1$, the economy's total endowment of good 1; its height is $\bar{\omega}_2$, the economy's total endowment of good 2. Any point in the box represents a (nonwasteful) division of the economy's total endowment between consumers 1 and 2. For example, Figure 15.B.1 depicts the endowment vector $\omega = ((\omega_{11}, \omega_{21}), (\omega_{12}, \omega_{22}))$ of the two consumers. Also depicted is another possible nonwasteful allocation, $x = ((x_{11}, x_{21}), (x_{12}, x_{22}))$; the fact that it is nonwasteful means that $(x_{12}, x_{22}) = (\bar{\omega}_1 - x_{11}, \bar{\omega}_2 - x_{21})$.

As is characteristic of general equilibrium theory, the wealth of a consumer is not given exogenously. Rather, for any prices $p = (p_1, p_2)$, consumer i's wealth equals the market value of his endowments of commodities, $p \cdot \omega_i = p_1 \omega_{1i} + p_2 \omega_{2i}$. Wealth levels are therefore determined by the values of prices. Hence, given the consumer's endowment vector ω_i, his budget set can be viewed solely as a

1. Alternatively, we could assume that each consumer (perhaps better called a *consumer type*) stands, not for an individual, but for a large number of identical consumers. This would make the price-taking assumption more plausible; and with equal numbers of the two types of consumers, the analysis in this section would be otherwise unaffected.

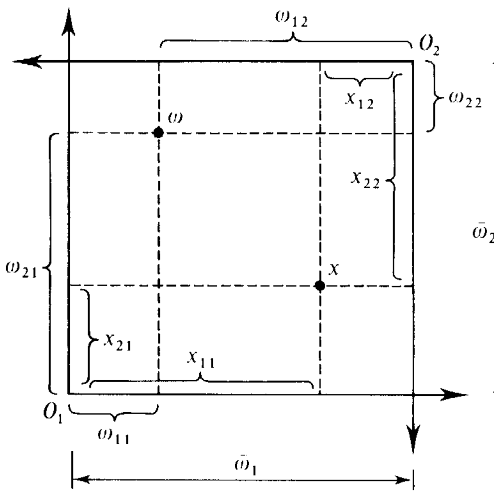


Figure 15.B.1 (left)
An Edgeworth box.

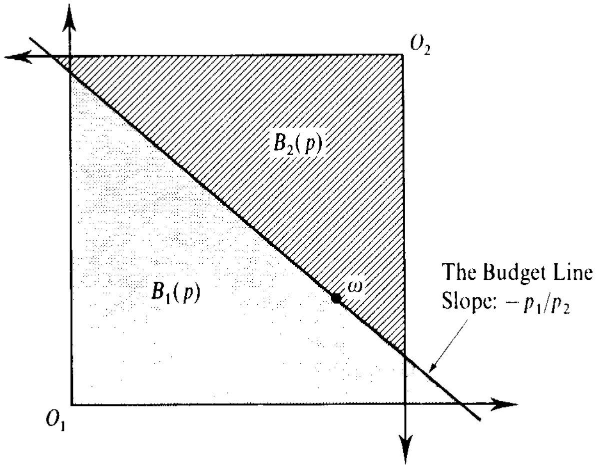


Figure 15.B.2 (right)
Budget sets.

function of prices:

$$B_i(p) = \{x_i \in \mathbb{R}^2_+ : p \cdot x_i \leq p \cdot \omega_i\}.$$

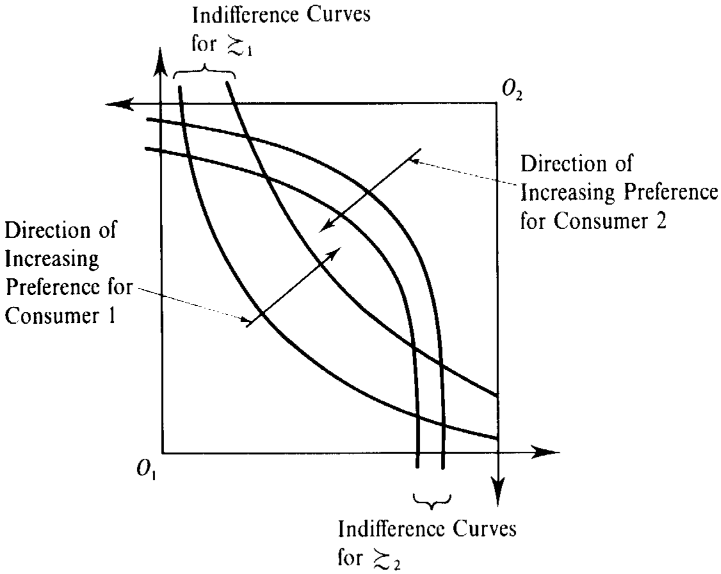
The budget sets of the two consumers can be represented in the Edgeworth box in a simple manner. To do so, we draw a line, known as the *budget line*, through the endowment point ω with slope $-(p_1/p_2)$, as shown in Figure 15.B.2. Consumer 1's budget set consists of all the nonnegative vectors below and to the left of this line (the shaded set). Consumer 2's budget set, on the other hand, consists of all the vectors above and to the right of this same line which give consumer 2 nonnegative consumption levels (the hatched set).[2] Observe that only allocations on the budget line are affordable to both consumers simultaneously at prices (p_1, p_2).[3]

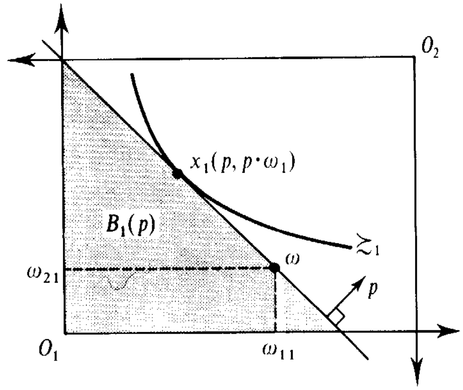
We can also depict the preferences $\succsim_i$ of each consumer i in the Edgeworth box, as in Figure 15.B.3. Except where otherwise noted, we assume that $\succsim_i$ is strictly convex, continuous, and strongly monotone (see Sections 3.B and 3.C for discussion of these conditions).

Figure 15.B.4 illustrates how the consumption vector demanded by consumer 1 can be determined for any price vector p. Given p, the consumer demands his most preferred point in $B_1(p)$, which can be expressed using his demand function as $x_1(p, p \cdot \omega_1)$ (this is the same demand function studied in Chapters 2 to 4; here wealth is $w_1 = p \cdot \omega_1$). In Figure 15.B.5, we see that as the price vector p varies, the budget line pivots around the endowment point ω, and the demanded consumptions trace out a curve, denoted by OC_1, that is called the *offer curve* of consumer 1. Note that this curve passes through the endowment point. Because at every p the endowment vector $\omega_1 = (\omega_{11}, \omega_{21})$ is affordable to consumer 1, it follows that this consumer must find every point on his offer curve at least as good as his endowment point.

2. Note, in particular, that the budget sets of the consumers may well extend outside the box.

3. There are other feasible allocations that are simultaneously affordable; but in these allocations some resources are not consumed by either consumer, and thus they cannot be depicted in an Edgeworth box. Because of the nonsatiation assumption to be made on preferences, we will not have to worry about such allocations.





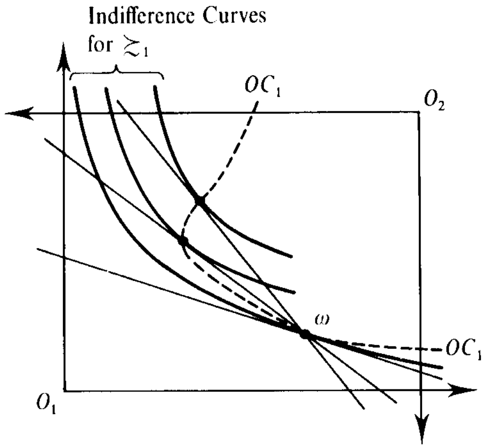


Figure 15.B.3 (top left)
Preferences in the Edgeworth box.

Figure 15.B.4 (top right)
Optimal consumption for consumer 1 at prices p.

Figure 15.B.5 (bottom)
Consumer 1's offer curve.

This implies that the consumer's offer curve lies within the upper contour set of ω_1 and that, if indifference curves are smooth, the offer curve must be tangent to the consumer's indifference curve at the endowment point.

Figure 15.B.6 represents the demanded bundles of the two consumers at some arbitrary price vector p. Note that the demands expressed by the two consumers are not compatible. The total demand for good 2 exceeds its total supply in the economy $\bar{\omega}_2$, whereas the total demand for good 1 is strictly less than its endowment $\bar{\omega}_1$. Put somewhat differently, consumer 1 is a *net demander* of good 2 in the sense that he wants to consume more than his endowment of that commodity. Although consumer 2 is willing to be a *net supplier* of that good (he wants to consume less than his endowment), he is not willing to supply enough to satisfy consumer 1's needs. Good 2 is therefore in *excess demand* in the situation depicted in the figure. In contrast, good 1 is in *excess supply*.

At a market equilibrium where consumers take prices as given, markets should clear. That is, the consumers should be able to fulfill their desired purchases and

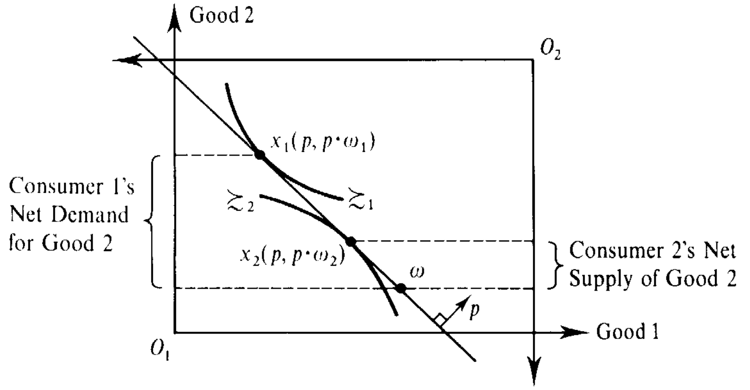


Figure 15.B.6
A price vector with excess demand for good 2 and excess supply for good 1.

sales of commodities at the going market prices. Thus, if one consumer wishes to be *a net demander* of some good, the other must be a *net supplier* of this good in exactly the same amount; that is, demand should equal supply. This gives us the notion of equilibrium presented in Definition 15.B.1.

Definition 15.B.1: A *Walrasian* (or *competitive*) *equilibrium* for an Edgeworth box economy is a price vector p^* and an allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$,

$$x_i^* \succsim_i x_i' \quad \text{for all} \quad x_i' \in B_i(p^*).$$

A Walrasian equilibrium is depicted in Figure 15.B.7. In Figure 15.B.7(a), we represent the equilibrium price vector p^* and the equilibrium allocation $x^* = (x_1^*, x_2^*)$. Each consumer i's demanded bundle at price vector p^* is x_i^*, and one consumer's net demand for a good is exactly matched by the other's net supply. Figure 15.B.7(b) adds to the depiction the consumers' offer curves and their indifference curves through ω. Note that at any equilibrium, the offer curves of the two consumers intersect. In fact, *any* intersection of the consumers' offer curves at an allocation different from the endowment point ω corresponds to an equilibrium because if $x^* = (x_1^*, x_2^*)$ is any such point of intersection, then x_i^* is the optimal consumption bundle for each consumer i for the budget line that goes through the two points ω and x^*.

In Figure 15.B.8, we show a Walrasian equilibrium where the equilibrium allocation lies on the boundary of the Edgeworth box. Once again, at price vector p^*, the two consumers' demands are compatible.

Note that each consumer's demand is homogeneous of degree zero in the price vector $p = (p_1, p_2)$; that is, if prices double, then the consumer's wealth also doubles and his budget set remains unchanged. Thus, from Definition 15.B.1, we see that if $p^* = (p_1^*, p_2^*)$ is a Walrasian equilibrium price vector, then so is $\alpha p^* = (\alpha p_1^*, \alpha p_2^*)$ for any $\alpha > 0$. In short, only the *relative* prices p_1^*/p_2^* are determined in an equilibrium.

Example 15.B.1: Suppose that each consumer i has the Cobb–Douglas utility function $u_i(x_{1i}, x_{2i}) = x_{1i}^{\alpha} x_{2i}^{1-\alpha}$. In addition, endowments are $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$. At prices $p = (p_1, p_2)$, consumer 1's wealth is $(p_1 + 2p_2)$ and therefore his demands lie on the offer curve (recall the derivation in Example 3.D.1):

$$OC_1(p) = \left(\frac{\alpha(p_1 + 2p_2)}{p_1}, \frac{(1-\alpha)(p_1 + 2p_2)}{p_2} \right).$$

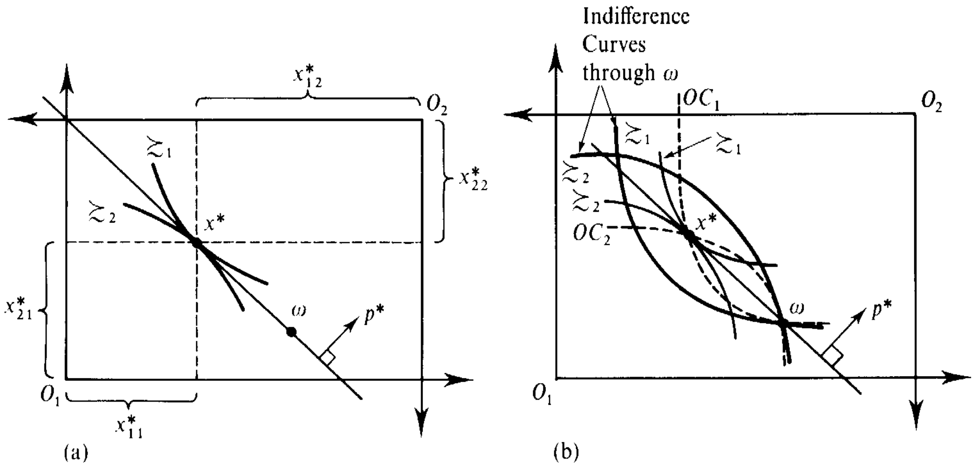


Figure 15.B.7 (top)
(a) A Walrasian equilibrium.
(b) The consumer's offer curves intersect at the Walrasian equilibrium allocation.

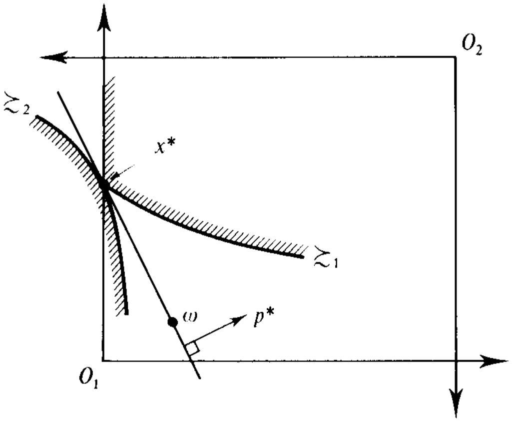


Figure 15.B.8 (bottom)
A Walrasian equilibrium allocation on the boundary of the Edgeworth box.

Observe that the demands for the first and the second good are, respectively, decreasing and increasing with p_1. This is how we have drawn OC_1 in Figure 15.B.7(b). Similarly, $OC_2(p) = (\alpha(2p_1 + p_2)/p_1, (1 - \alpha)(2p_1 + p_2)/p_2)$. To determine the Walrasian equilibrium prices, note that at these prices the total amount of good 1 consumed by the two consumers must equal 3 $(= \omega_{11} + \omega_{12})$. Thus,

$$\frac{\alpha(p_1^* + 2p_2^*)}{p_1^*} + \frac{\alpha(2p_1^* + p_2^*)}{p_1^*} = 3.$$

Solving this equation yields

$$\frac{p_1^*}{p_2^*} = \frac{\alpha}{1 - \alpha}. \tag{15.B.2}$$

Observe that at any prices (p_1^*, p_2^*) satisfying condition (15.B.2), the market for good 2 clears as well (you should verify this). This is a general feature of an Edgeworth box economy: To determine equilibrium prices we need only determine prices at which one of the markets clears; the other market will necessarily clear at these prices. This point can be seen graphically in the Edgeworth box: Because both consumers' demanded bundles lie on the same budget line, if the amounts of commodity 1 demanded are compatible, then so must be those for commodity 2. (See also Exercise 15.B.1.) ■

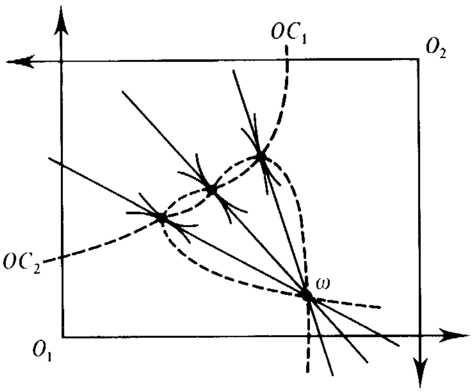


Figure 15.B.9
Multiple Walrasian equilibria.

The Edgeworth box, simple as it is, is remarkably powerful. There are virtually no phenomena or properties of general equilibrium exchange economies that cannot be depicted in it. Consider, for example, the issue of the uniqueness of Walrasian equilibrium. In Chapter 10, we saw that if there is a numeraire commodity relative to which preferences admit a quasilinear representation, then (with strict convexity of preferences) the equilibrium consumption allocation and relative prices are unique. In Figure 15.B.7, we also have uniqueness (see Exercise 15.B.2 for a more explicit discussion). Yet, as the Edgeworth box in Figure 15.B.9 shows, this property does not generalize. In that figure, preferences (which are entirely nonpathological) are such that the offer curves change curvature and interlace several times. In particular, they intersect for prices such that p_1/p_2 is equal to $\frac{1}{2}$, 1, and 2. For the sake of completeness, we present an analytical example with the features of the figure.

Example 15.B.2: Let the two consumers have utility functions

$$u_1(x_{11}, x_{21}) = x_{11} - \tfrac{1}{8}x_{21}^{-8} \text{ and } u_2(x_{12}, x_{22}) = -\tfrac{1}{8}x_{12}^{-8} + x_{22}.$$

Note that the utility functions are quasilinear (which, in particular, facilitates the computation of demand), but with respect to *different* numeraires. The endowments are $\omega_1 = (2, r)$ and $\omega_2 = (r, 2)$, where r is chosen to guarantee that the equilibrium prices turn out to be round numbers. Precisely, $r = 2^{8/9} - 2^{1/9} > 0$. In Exercise 15.B.5, you are asked to compute the offer curves of the two consumers. They are:

$$OC_1(p_1, p_2) = \left(2 + r\left(\frac{p_2}{p_1}\right) - \left(\frac{p_2}{p_1}\right)^{8/9}, \left(\frac{p_2}{p_1}\right)^{-1/9}\right) \gg 0$$

and

$$OC_2(p_1, p_2) = \left(\left(\frac{p_1}{p_2}\right)^{-1/9}, 2 + r\left(\frac{p_1}{p_2}\right) - \left(\frac{p_1}{p_2}\right)^{8/9}\right) \gg 0.$$

Note that, as illustrated in Figure 15.B.9, and in contrast with Example 15.B.1, consumer 1's demand for good 1 (and symmetrically for consumer 2) may be increasing in p_1.

To compute the equilibria it is sufficient to solve the equation that equates the total demand of the second good to its total supply, or

$$\left(\frac{p_2}{p_1}\right)^{-1/9} + 2 + r\left(\frac{p_1}{p_2}\right) - \left(\frac{p_1}{p_2}\right)^{8/9} = 2 + r.$$

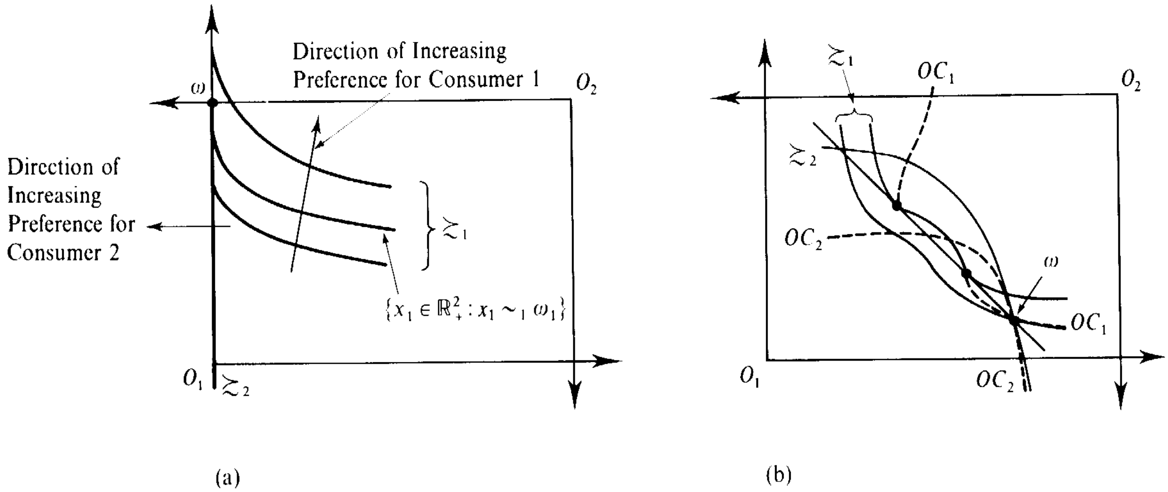


Figure 15.B.10 (a) and (b): Two examples of nonexistence of Walrasian equilibrium.

Recalling the value of r, this equation has three solutions for p_1/p_2: 2, 1, and $\frac{1}{2}$ (you should check this). ■

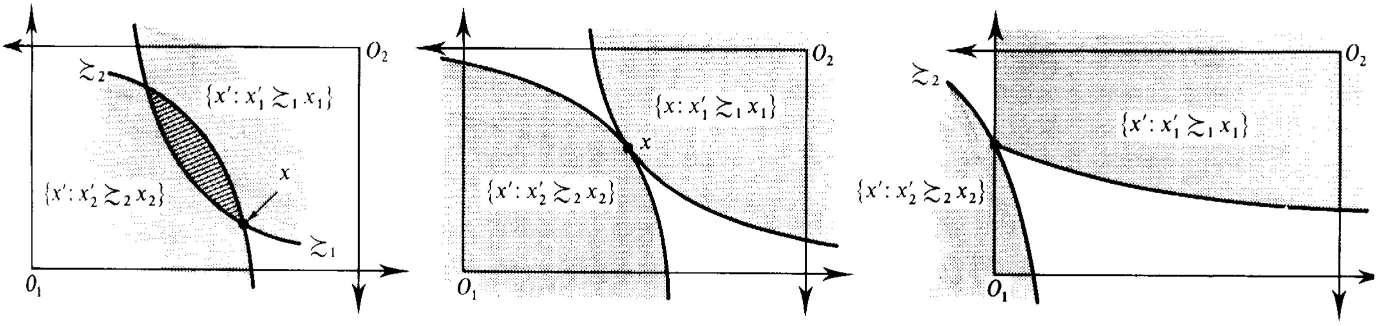
It may also happen that a pure exchange economy does not have *any* Walrasian equilibria. For example, Figure 15.B.10(a) depicts a situation in which the endowment lies on the boundary of the Edgeworth box (in the northwest corner). Consumer 2 has all the endowment of good 1 and desires only good 1. Consumer 1 has all the endowment of good 2 and his indifference set containing ω_1, $\{x_1 \in \mathbb{R}^2_+ : x_1 \sim_1 \omega_1\}$, has an infinite slope at ω_1 (note, however, that at ω_1, consumer 1 would strictly prefer receiving more of good 1). In this situation, there is no price vector p^* at which the consumers' demands are compatible. If $p_2/p_1 > 0$ then consumer 2 optimal demand is to keep his initial bundle ω_2, whereas the initial bundle ω_1 is never consumer 1's optimal demand (no matter how large the relative price of the first good, consumer 1 always wishes to buy a strictly positive amount of it). On the other hand, consumer 1's demand for good 2 is infinite when $p_2/p_1 = 0$. Note for future reference that consumer 2's preferences in this example are not strongly monotone.

Figure 15.B.10(b) depicts a second example of nonexistence. There, consumer 1's preferences are nonconvex. As a result, consumer 1's offer curve is disconnected, and there is no crossing point of the two consumers' offer curves (other than the endowment point, which is not an equilibrium allocation here).

In Chapter 17, we will study the conditions under which the existence of a Walrasian equilibrium is assured.

Welfare Properties of Walrasian Equilibria

A central question in economic theory concerns the welfare properties of equilibria. Here we shall focus on the notion of Pareto optimality, which we have already encountered in Chapter 10 (see, in particular, Section 10.B). An economic outcome is *Pareto optimal* (or *Pareto efficient*) if there is no alternative feasible outcome at which every individual in the economy is at least as well off and some individual is



(a) (b) (c)

Figure 15.B.11 (a) Allocation x is not Pareto optimal. (b) Allocation x is Pareto optimal. (c) Allocation x is Pareto optimal.

strictly better off. Definition 15.B.2 expresses this idea in the setting of our two-consumer, pure exchange economy.

Definition 15.B.2: An allocation x in the Edgeworth box is *Pareto optimal* (or *Pareto efficient*) if there is no other allocaton x' in the Edgeworth box with $x'_i \succsim_i x_i$ for $i = 1, 2$ and $x'_i \succ_i x_i$ for some i.

Figure 15.B.11(a) depicts an allocation x that is not Pareto optimal. Any allocation in the interior of the crosshatched region of the figure, the intersection of the sets $\{x'_1 \in \mathbb{R}^2_+ : x'_1 \succsim_1 x_1\}$ and $\{x'_2 \in \mathbb{R}^2_+ : x'_2 \succsim_2 x_2\}$ within the Edgeworth box, is a feasible allocation that makes both consumers strictly better off than at x. The allocation x depicted in Figure 15.B.11(b), on the other hand, is Pareto optimal because the intersection of the sets $\{x'_i \in \mathbb{R}^2_+ : x'_i \succsim_i x_i\}$ for $i = 1, 2$ consists only of the point x. Note that if a Pareto optimal allocation x is an interior point of the Edgeworth box, then the consumers' indifference curves through x must be tangent (assuming that they are smooth). Figure 15.B.11(c) depicts a Pareto optimal allocation x that is not interior; at such a point, tangency need not hold.

The set of all Pareto optimal allocations is known as the *Pareto set*. An example is illustrated in Figure 15.B.12. The figure also displays the *contract curve*, the part of the

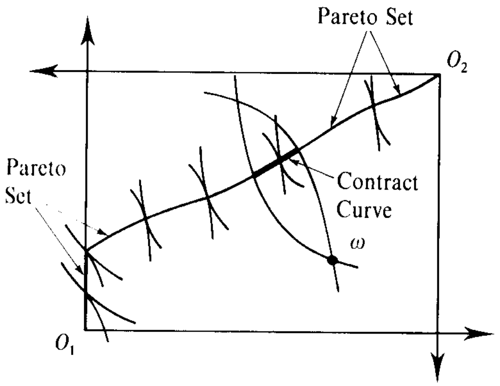


Figure 15.B.12
The Pareto set and the contract curve.

Pareto set where both consumers do at least as well as at their initial endowments. The reason for this term is that we might expect any bargaining between the two consumers to result in an agreement to trade to some point on the contract curve; these are the only points at which both of them do as well as at their initial endowments and for which there is no alternative trade that can make both consumers better off.

We can now verify a simple but important fact: *Any Walrasian equilibrium allocation x^* necessarily belongs to the Pareto set.* To see this, note that by the definition of a Walrasian equilibrium the budget line separates the two at-least-as-good-as sets associated with the equilibrium allocation, as seen in Figures 15.B.7(a) and 15.B.8. The only point in common between these two sets is x^* itself. Thus, at any competitive allocation x^*, there is no alternative feasible allocation that can benefit one consumer without hurting the other. The conclusion that Walrasian allocations yield Pareto optimal allocations is an expression of the *first fundamental theorem of welfare economics*, a result that, as we shall see in Chapter 16, holds with great generality. Note, moreover, that since each consumer must be at least as well off in a Walrasian equilibrium as by simply consuming his endowment, any Walrasian equilibrium lies in the contract curve portion of the Pareto set.

The first fundamental welfare theorem provides, for competitive market economies, a formal expression of Adam Smith's "invisible hand." Under perfectly competitive conditions, any equilibrium allocation is a Pareto optimum, and the only possible welfare justification for intervention in the economy is the fulfillment of distributional objectives.

The *second fundamental theorem of welfare economics*, which we also discuss extensively in Chapter 16, offers a (partial) converse result. Roughly put, it says that *under convexity assumptions (not required for the first welfare theorem), a planner can achieve any desired Pareto optimal allocation by appropriately redistributing wealth in a lump-sum fashion and then "letting the market work."* Thus, the second welfare theorem provides a theoretical affirmation for the use of competitive markets in pursuing distributional objectives.

Definition 15.B.3 is a more formal statement of the concept of an equilibrium with lump-sum wealth redistribution.

Definition 15.B.3: An allocation x^* in the Edgeworth box is supportable as an *equilibrium with transfers* if there is a price system p^* and wealth transfers T_1 and T_2 satisfying $T_1 + T_2 = 0$, such that for each consumer i we have

$$x_i^* \succsim_i x_i' \quad \text{for all } x_i' \in \mathbb{R}_+^2 \quad \text{such that} \quad p^* \cdot x_i' \leq p^* \cdot \omega_i + T_i.$$

Note that the transfers sum to zero in Definition 15.B.3; the planner runs a balanced budget, merely redistributing wealth between the consumers.

Equipped with Definition 15.B.3, we can state more formally a version of the second welfare theorem as follows: if the preferences of the two consumers in the Edgeworth box are continuous, convex, and strongly monotone, then *any Pareto optimal allocation is supportable as an equilibrium with transfers.* This result is illustrated in Figure 15.B.13(a), where the consumer's endowments are at point ω. Suppose that for distributional reasons, the socially desired allocation is the Pareto optimal allocation x^*. Then if a tax authority constructs a transfer of wealth between the two consumers that shifts the budget line to the location

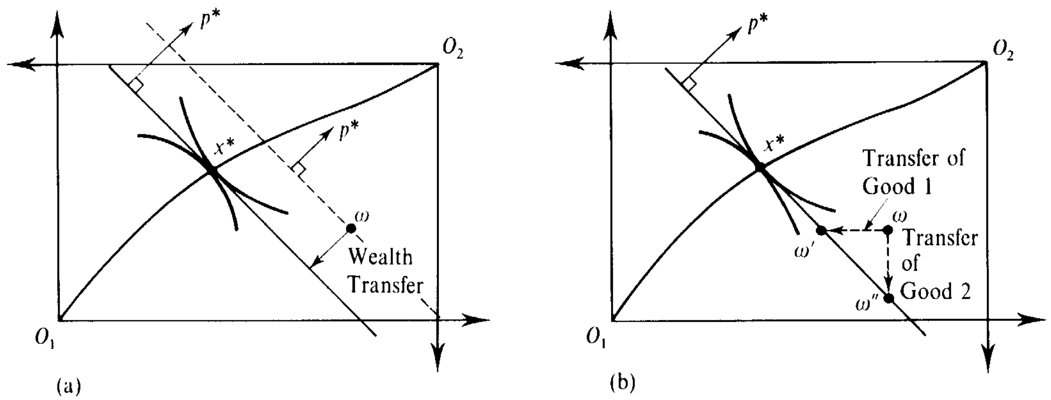


Figure 15.B.13 The second fundamental welfare theorem. (a) Using wealth transfers. (b) Using transfers of endowments.

indicated in the figure, the price vector p^* clears the markets for the two goods, and allocation x^* results.

Note that this wealth transfer may also be accomplished by directly transferring endowments. As Figure 15.B.13(b) illustrates, a transfer of good 1 that moves the endowment vector to ω' will have the price vector p^* and allocation x^* as a Walrasian equilibrium. A transfer of good 2 that changes endowments to ω'' does so as well. In fact, if all commodities can be easily transferred, then we could equally well move the endowment vector directly to allocation x^*. From this new endowment point, the Walrasian equilibrium involves no trade.[4]

Figure 15.B.14 shows that the second welfare theorem may fail to hold when preferences are not convex. In the figure, $x^* = (x_1^*, x_2^*)$ is a Pareto optimal allocation that is not supportable as an equilibrium with transfers. At the budget line with the property that consumer 2 would demand x_2^*, consumer 1 would prefer a point other than x_1^* (such as x'_1). Convexity, as it turns out, is a critical assumption for the second welfare theorem.

A failure of the second welfare theorem of a different kind is illustrated in Figure 15.B.10(a). There, the initial endowment allocation ω is a Pareto optimal allocation, but it cannot be supported as an equilibrium with transfers (you should check this). In this case, it is the assumption that consumers' preferences are strongly monotone that is violated.

For further illustrations of Edgeworth box economies see, for example, Newman (1965).

15.C The One-Consumer, One-Producer Economy

We now introduce the possibility of production. To do so in the simplest-possible setting, we suppose that there are two price-taking economic agents, a single

4. In practice, endowments may be difficult to transfer (e.g., human capital), and so the ability to use wealth transfers (or transfers of only a limited number of commodities) may be important. It is worth observing that there is one attractive feature of transferring endowments directly to the desired Pareto optimal allocation: we can be assured that x^* is the *unique* Walrasian equilibrium allocation after the transfers (strictly speaking this requires a strict convexity assumption on preferences).

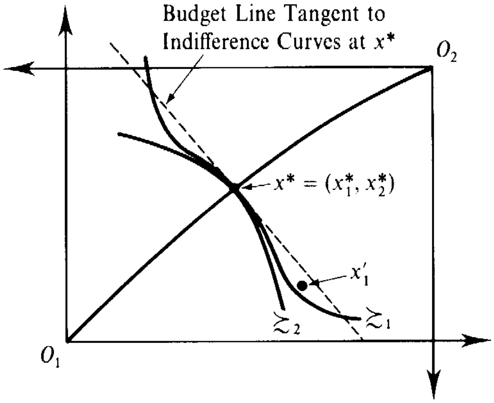


Figure 15.B.14
Failure of the second welfare theorem with nonconvex preferences.

consumer and a single firm, and two goods, the labor (or leisure) of the consumer and a consumption good produced by the firm.[5]

The consumer has continuous, convex, and strongly monotone preferences $\succsim$ defined over his consumption of leisure x_1 and the consumption good x_2. He has an endowment of $\bar{L}$ units of leisure (e.g., 24 hours in a day) and no endowment of the consumption good.

The firm uses labor to produce the consumption good according to the increasing and strictly concave production function $f(z)$, where z is the firm's labor input. Thus, to produce output, the firm must hire the consumer, effectively purchasing some of the consumer's leisure from him. We assume that the firm seeks to maximize its profits taking market prices as given. Letting p be the price of its output and w be the price of labor, the firm solves

$$\operatorname*{Max}_{z \geq 0} \quad pf(z) - wz. \tag{15.C.1}$$

Given prices (p, w), the firm's optimal labor demand is $z(p, w)$, its output is $q(p, w)$, and its profits are $\pi(p, w)$.

As we noted in Chapter 5, firms are owned by consumers. Thus, we assume that the consumer is the sole owner of the firm and receives the profits earned by the firm $\pi(p, w)$. As with the price-taking assumption, the idea of the consumer being hired by his own firm through an anonymous labor market may appear strange in this model with only two agents. Nevertheless, bear with us; our aim is to illustrate the workings of more complicated many-consumer general equilibrium models in the simplest-possible way.[6]

Letting $u(x_1, x_2)$ be a utility function representing $\succsim$, the consumer's problem given prices (p, w) is

$$\operatorname*{Max}_{(x_1, x_2) \in \mathbb{R}^2_+} \quad u(x_1, x_2) \tag{15.C.2}$$
$$\text{s.t.} \quad px_2 \leq w(\bar{L} - x_1) + \pi(p, w).$$

5. One-consumer economies are sometimes referred to as *Robinson Crusoe economies*.

6. The point made in footnote 1 can be repeated here: we could imagine that the firm and the consumer stand for a large number of identical firms and consumers. We comment a bit more on this interpretation at the end of this Section.

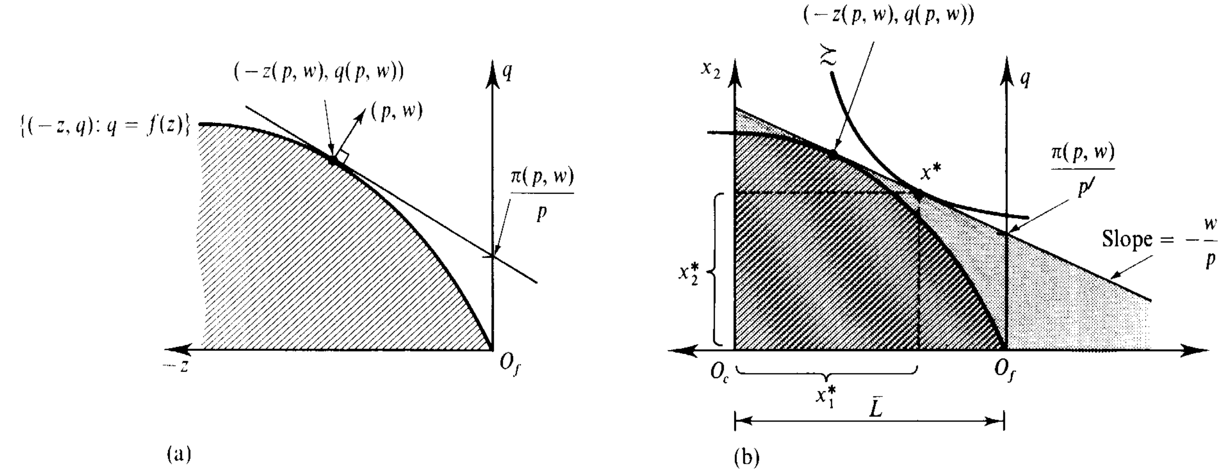


Figure 15.C.1 (a) The firm's problem. (b) The consumer's problem.

The budget constraint in (15.C.2) reflects the two sources of the consumer's purchasing power: If the consumer supplies an amount $(\bar{L} - x_1)$ of labor when prices are (p, w), then the total amount he can spend on the consumption good is his labor earnings $w(\bar{L} - x_1)$ plus the profit distribution from the firm $\pi(p, w)$. The consumer's optimal demands in problem (15.C.2) for prices (p, w) are denoted by $(x_1(p, w), x_2(p, w))$.

A Walrasian equilibrium in this economy involves a price vector (p^*, w^*) at which the consumption and labor markets clear; that is, at which

$$x_2(p^*, w^*) = q(p^*, w^*) \tag{15.C.3}$$

and

$$z(p^*, w^*) = \bar{L} - x_1(p^*, w^*) \tag{15.C.4}$$

Figure 15.C.1 illustrates the working of this one-consumer, one-firm economy. Figure 15.C.1(a) depicts the firm's problem. As in Chapter 5, we measure the firm's use of labor input on the horizontal axis as a negative quantity. Its output is depicted on the vertical axis. The production set associated with the production function $f(z)$ is also shown, as are the profit-maximizing input and output levels at prices (p, w), $z(p, w)$ and $q(p, w)$, respectively.

Figure 15.C.1(b) adapts this diagram to represent the consumer's problem. Leisure and consumption levels are measured from the origin denoted O_c at the lower-left-hand corner of the diagram, which is determined by letting the length of the segment $[O_c, O_f]$ be equal to $\bar{L}$, the total labor endowment. The figure depicts the consumer's (shaded) budget set given prices (p, w) and profits $\pi(p, w)$. Note that if the consumer consumes $\bar{L}$ units of leisure then since he sells no labor, he can purchase $\pi(p, w)/p$ units of the consumption good. Thus, the budget line must cut the vertical q-axis at height $\pi(p, w)/p$. In addition, for each unit of labor he sells, the consumer earns w and can therefore afford to purchase w/p units of x_2. Hence, the budget line has slope $-(w/p)$. Observe that the consumer's budget line is exactly the isoprofit line associated with the solution to the firm's profit-maximization problem in Figure 15.C.1(a), that is, the set of points $\{(-z, q): pq - wz = \pi(p, w)\}$ that yield profits of $\pi(p, w)$.

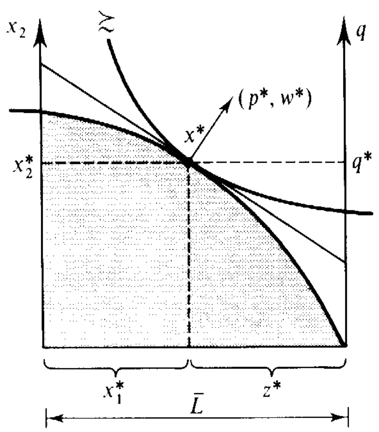


Figure 15.C.2
A Walrasian equilibrium.

The prices depicted in Figure 15.C.1(b) are not equilibrium prices; at these prices, there is an excess demand for labor (the firm wants more labor than the consumer is willing to supply) and an excess supply of the produced good. An equilibrium price vector (p^*, w^*) that clears the markets for the two goods is depicted in Figure 15.C.2.

There is a very important fact to notice from Figure 15.C.2: *A particular consumption leisure combination can arise in a competitive equilibrium if and only if it maximizes the consumer's utility subject to the economy's technological and endowment constraints.* That is, the Walrasian equilibrium allocation is the same allocation that would be obtained if a planner ran the economy in a manner that maximized the consumer's well-being. Thus, we see here an expression of the fundamental theorems of welfare economics: Any Walrasian equilibrium is Pareto optimal, and the Pareto optimal allocation is supportable as a Walrasian equilibrium.[7]

The indispensability of convexity for the second welfare theorem can again be observed in Figure 15.C.3(a). There, the allocation x^* maximizes the welfare of the consumer, but for the only value of relative prices that could support x^* as a utility-maximizing bundle, the firm does not maximize profits even locally (i.e., at the relative prices w/p, there are productions arbitrarily close to x^* yielding higher profits). In contrast, the first welfare theorem remains applicable even in the presence of nonconvexities. As Figure 15.C.3(b) suggests, any Walrasian equilibrium maximizes the well-being of the consumer in the feasible production set.

Under certain circumstances, the model studied in this section can be rigorously justified as representing the outcome of a more general economy by interpreting the "firm" as a representative producer (see Section 5.E) and the "consumer" as a representative consumer (see Section 4.D). The former is always possible, but the latter—that is, the existence of a (normative) representative consumer—requires strong conditions. If, however, the economy

7. In a single-consumer economy, the test for Pareto optimality reduces to the question of whether the well-being of the single consumer is being maximized (subject to feasibility constraints). Note that given the convexity of preferences and the strict convexity of the aggregate production set assumed here, there is a unique Pareto optimal consumption vector (and therefore a unique equilibrium).

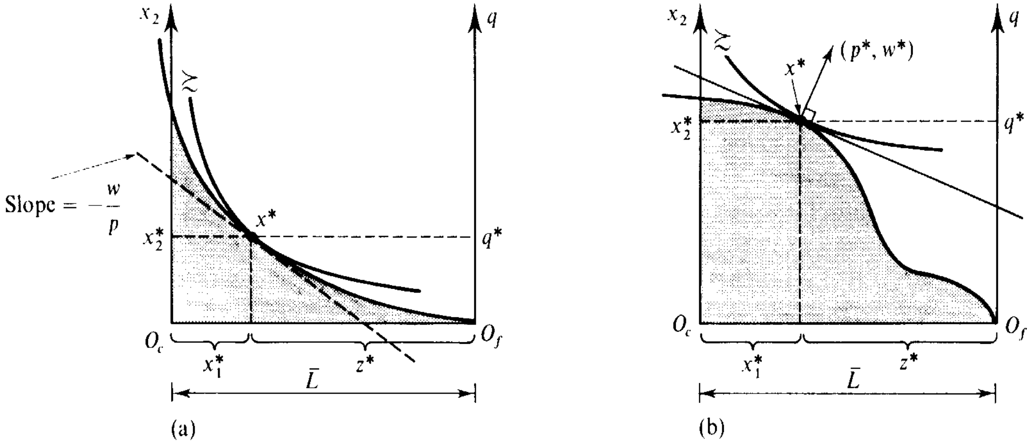


Figure 15.C.3 (a) Failure of the second welfare theorem with a nonconvex technology. (b) The first welfare theorem applies even with a nonconvex technology.

is composed of many consumers with identical concave utility functions and identical initial endowments, and if society has a strictly concave social welfare function in which these consumers are treated symmetrically, then a (normative) representative consumer exists who has the same utility function as the consumers over levels of per capita consumption.[8] (We can also think of the representative firm's input and output choices as being on a per capita basis). For more general conditions under which a representative consumer exists, see Section 4.D.

15.D The 2 × 2 Production Model

In this section, we discuss an example that concentrates on general equilibrium effects in production.

To begin, consider an economy in which the production sector consists of J firms. Each firm j produces a consumer good q_j directly from a vector of L primary (i.e., nonproduced) inputs, or *factors*, $z_j = (z_{1j}, \ldots, z_{Lj}) \geq 0$.[9] Firm j's production takes place by means of a concave, strictly increasing, and differentiable production function $f_j(z_j)$. Note that there are no intermediate goods (i.e., produced goods that are used as inputs). The economy has total endowments of the L factor inputs, $(\bar{z}_1, \ldots, \bar{z}_L) \gg 0$. These endowments are initially owned by consumers and have a use only as production inputs (i.e., consumers do not wish to consume them).

To concentrate on the factor markets of the economy, we suppose that the prices of the J produced consumption goods are fixed at $p = (p_1, \ldots, p_J)$, The leading example for this assumption is that of a small open economy whose trading decisions in the world markets for consumption goods have little effect on the world prices of

8. To see this, note that an equal distribution of wealth (which is what occurs here in the absence of any wealth transfers given the identical endowments of the consumers) maximizes social welfare for any price vector and aggregate wealth level.

9. Some of these outputs may be the same good; that is, firms j and j' may produce the same commodity.

these goods.[10] Output is sold in world markets. Factors, on the other hand, are immobile and must be used for production within the country.

The central question for our analysis concerns the equilibrium in the factor markets; that is, we wish to determine the equilibrium factor prices $w = (w_1, \ldots, w_L)$ and the allocation of the economy's factor endowments among the J firms.[11]

Given output prices $p = (p_1, \ldots, p_J)$ and input prices $w = (w_1, \ldots, w_L)$, a profit-maximizing production plan for firm j solves

$$\underset{z_j \geq 0}{\text{Max}} \quad p_j f_j(z_j) - w \cdot z_j.$$

We denote firm j's set of optimal input demands given prices (p, w) by $z(p, w) \subset \mathbb{R}^L_+$. Because consumers have no direct use for their factor endowments, the total factor supply will be $(\bar{z}_1, \ldots, \bar{z}_L)$ as long as the input prices w_ℓ are strictly positive (the only case that will concern us here). An equilibrium for the factor markets of this economy given the fixed output prices p therefore consists of an input price vector $w^* = (w_1^*, \ldots, w_L^*) \gg 0$ and a factor allocation

$$(z_1^*, \ldots, z_J^*) = ((z_{11}^*, \ldots, z_{L1}^*), \ldots, (z_{1J}^*, \ldots, z_{LJ}^*)),$$

such that firms receive their desired factor demands under prices (p, w^*) and all the factor markets clear, that is, such that

$$z_j^* \in z_j(p, w) \qquad \text{for all } j = 1, \ldots, J$$

and

$$\sum_j z_{\ell j}^* = \bar{z}_\ell \qquad \text{for all } \ell = 1, \ldots, L.$$

Because of the concavity of firms' production functions, first-order conditions are both necessary and sufficient for the characterization of optimal factor demands. Therefore, the $L(J + 1)$ variables formed by the factor allocation $(z_1^*, \ldots, z_J^*) \in \mathbb{R}^{LJ}_+$ and the factor prices $w^* = (w_1^*, \ldots, w_L^*)$ constitute an equilibrium if and only if they satisfy the following $L(J + 1)$ equations (we assume an interior solution here):

$$p_j \frac{\partial f_j(z_j^*)}{\partial z_{\ell j}} = w_\ell^* \qquad \text{for } j = 1, \ldots, J \text{ and } \ell = 1, \ldots, L \tag{15.D.1}$$

and

$$\sum_j z_{\ell j}^* = \bar{z}_\ell \qquad \text{for } \ell = 1, \ldots, L. \tag{15.D.2}$$

The equilibrium output levels are then $q_j^* = f_j(z_j^*)$ for every j.

Equilibrium conditions for *outputs* and factor prices can alternatively be stated using the firms' cost functions $c_j(w, q_j)$ for $j = 1, \ldots, J$. Output levels $(q_1^*, \ldots, q_J^*) \gg 0$ and factor prices $w^* \gg 0$ constitute an equilibrium if and only if the following

10. See Exercise 15.D.4 for an endogenous determination (up to a scalar multiple) of the prices $p = (p_1, \ldots, p_J)$.

11. Note that once the factor prices and allocations are determined, each consumer's demands can be readily determined from his demand function given the exogenous prices $(p_1, \ldots, p_J)$ and the wealth derived from factor input sales and profit distributions. Recall that the current model is completed by assuming that this demand is met in the world markets.

conditions hold:

$$p_j = \frac{\partial c_j(w^*, q_j^*)}{\partial q_j} \qquad \text{for } j = 1, \ldots, J, \qquad (15.D.3)$$

$$\sum_j \frac{\partial c_j(w^*, q_j^*)}{\partial w_\ell} = \bar{z}_\ell \qquad \text{for } \ell = 1, \ldots, L. \qquad (15.D.4)$$

Conditions (15.D.3) and (15.D.4) constitute a system of $L + J$ equations in the $L + J$ endogenous variables $(w_1, \ldots, w_L)$ and $(q_1, \ldots, q_J)$. Condition (15.D.3) states that each firm must be at a profit-maximizing output level given prices p and w^*. If so, firm j's optimal demand for the ℓth input is $z_{\ell j}^* = \partial c_j(w, q_j^*)/\partial w_\ell$ (this is Shepard's lemma; see Proposition 5.C.2). Condition (15.D.4) is therefore the factor market-clearing condition.

Before examining the determinants of the equilibrium factor allocation in greater detail, we note that the equilibrium factor allocation $(z_1^*, \ldots, z_J^*)$ in this model is exactly the factor allocation that would be chosen by a revenue-maximizing planner, thus providing us with yet another expression of the welfare-maximizing property of competitive allocations (the first welfare theorem).[12] To see this, consider the problem faced by a planning authority who is charged with coordinating factor allocations for the economy in order to maximize the gross revenues from the economy's production activities:

$$\underset{(z_1,\ldots,z_J)\geq 0}{\text{Max}} \quad \sum_j p_j f_j(z_j) \qquad (15.D.5)$$
$$\text{s.t.} \quad \textstyle\sum_j z_j = \bar{z}.$$

How does the equilibrium factor allocation $(z_1^*, \ldots, z_J^*)$ compare with what this planner does? Recall from Section 5.E that whenever we have a collection of J price-taking firms, their profit-maximizing behavior is compatible with the behavior we would observe if the firms were to maximize their profits jointly taking the prices of outputs and factors as given. That is, the factor demands $(z_1^*, \ldots, z_J^*)$ solve

$$\underset{(z_1,\ldots,z_J)\geq 0}{\text{Max}} \quad \sum_j (p_j f_j(z_j) - w^* \cdot z_j). \qquad (15.D.6)$$

Since $\sum_j z_j^* = \bar{z}$ (by the equilibrium property of market clearing), the factor demands $(z_1^*, \ldots, z_J^*)$ must also solve problem (15.D.6) subject to the further constraint that $\sum_j z_j = \bar{z}$. But this implies that the factor demands $(z_1^*, \ldots, z_J^*)$ in fact solve problem (15.D.5): if we must have $\sum_j z_j = \bar{z}$, then the total cost $w^* \cdot (\sum_j z_j)$ is given, and so the joint profit-maximizing problem (15.D.6) reduces to the revenue-maximizing problem (15.D.5).

One benefit of the property just established is that it can be used to obtain the equilibrium factor allocation without a previous explicit computation of the equilibrium factor prices; we simply need to solve problem (15.D.5) directly. It also provides a useful way of viewing the equilibrium factor prices. To see this, consider again the joint profit-maximization problem (15.D.6). We can approach this problem in an equivalent manner by first deriving an aggregate

12. Note that maximization of economy-wide revenue from production would be the goal of any planner who wanted to maximize consumer welfare: it allows for the maximal purchases of consumption goods, at the fixed world prices.

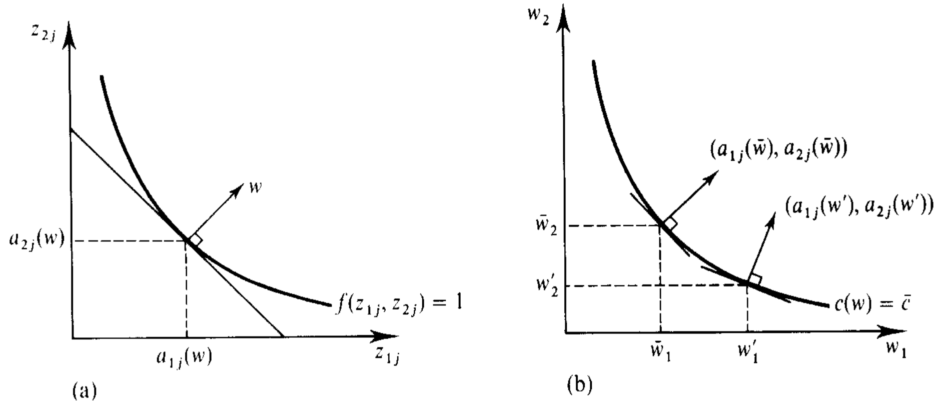


Figure 15.D.1
(a) A unit isoquant.
(b) The unit cost function.

production function for dollars:

$$f(z) = \underset{(z_1,\dots,z_J)\geq 0}{\text{Max}} \quad p_1 f_1(z_1) + \cdots + p_J f_J(z_J)$$
$$\text{s.t. } \textstyle\sum_j z_j = z.$$

The aggregate factor demands must then solve $\text{Max}_{z\geq 0}(f(z) - w \cdot z)$. For every ℓ, the first-order condition for this problem is $w_\ell = \partial f(z)/\partial z_\ell$. Moreover, at an equilibrium, the aggregate usage of factor ℓ must be exactly $\bar{z}_\ell$. Hence, the equilibrium factor price of factor ℓ must be $w_\ell = \partial f(\bar{z})/\partial z_\ell$; that is, *the price of factor ℓ must be exactly equal to its aggregate marginal productivity (in terms of revenue)*. Since $f(\cdot)$ is concave, this observation by itself generates some interesting comparative statics. For example, a change in the endowment of a single input must change the equilibrium price of the input in the opposite direction.

Let us now be more specific and take $J = L = 2$, so that the economy under study produces two outputs from two primary factors. We also assume that the production functions $f_1(z_{11}, z_{21})$, $f_2(z_{12}, z_{22})$ are homogeneous of degree one (so the technologies exhibit constant returns to scale; see Section 5.B). This model is known as the *2 × 2 production model*. In applications, factor 1 is often thought of as labor and factor 2 as capital.

For every vector of factor prices $w = (w_1, w_2)$, we denote by $c_j(w)$ the minimum cost of producing one unit of good j and by $a_j(w) = (a_{1j}(w), a_{2j}(w))$ the input combination (assumed unique) at which this minimum cost is reached. Recall again from Proposition 5.C.2 that $\nabla c_j(w) = (a_{1j}(w), a_{2j}(w))$.

Figure 15.D.1(a) depicts the unit isoquant of firm j,

$$\{(z_{1j}, z_{2j}) \in \mathbb{R}^2_+ : f_j(z_{1j}, z_{2j}) = 1\},$$

along with the cost-minimizing input combination $(a_{1j}(w), a_{2j}(w))$. In Figure 15.D.1(b), we draw a level curve of the unit cost function, $\{(w_1, w_2)\colon c_j(w_1, w_2) = \bar{c}\}$. This curve is downward sloping because as w_1 increases, w_2 must fall in order to keep the minimized costs of producing one unit of good j unchanged. Moreover, the set $\{(w_1, w_2)\colon c_j(w_1, w_2) \geq \bar{c}\}$ is convex because of the concavity of the cost function $c_j(w)$ in w. Note that the vector $\nabla c_j(\bar{w})$, which is normal to the level curve at $\bar{w} = (\bar{w}_1, \bar{w}_2)$, is exactly $(a_{1j}(\bar{w}), a_{2j}(\bar{w}))$. As we move along the curve toward higher w_1 and lower w_2, the ratio $a_{1j}(w)/a_{2j}(w)$ falls.

Consider, first, the efficient factor allocations for this model. In Figure 15.D.2, we

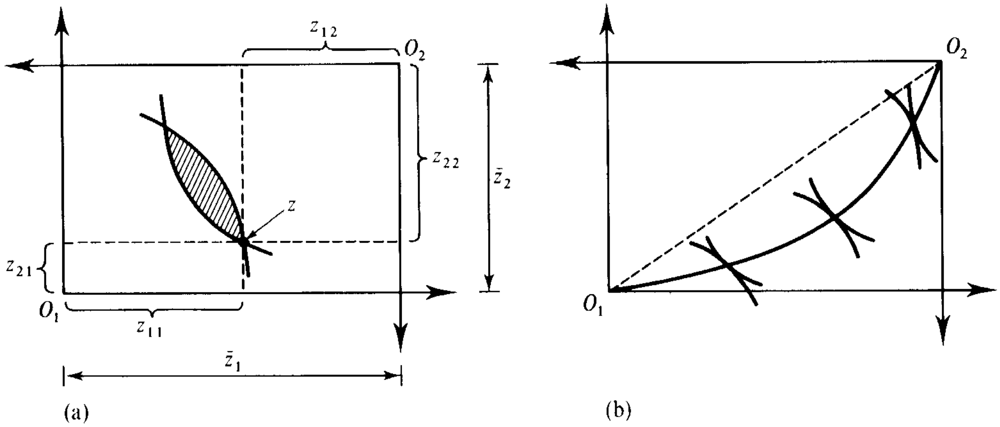


Figure 15.D.2 (a) An inefficient factor allocation. (b) The Pareto set of factor allocations.

represent the possible allocations of the factor endowments between the two firms in an Edgeworth box of size $\bar{z}_1$ by $\bar{z}_2$. The factors used by firm 1 are measured from the southwest corner; those used by firm 2 are measured from the northeast corner. We also represent the isoquants of the two firms in this Edgeworth box. Figure 15.D.2(a) depicts an inefficient allocation z of the inputs between the two firms: Any allocation in the interior of the hatched region generates more output of *both* goods than does z. Figure 15.D.2(b), on the other hand, depicts the Pareto set of factor allocations, that is, the set of factor allocations at which it is not possible, with the given total factor endowments, to produce more of one good without producing less of the other.

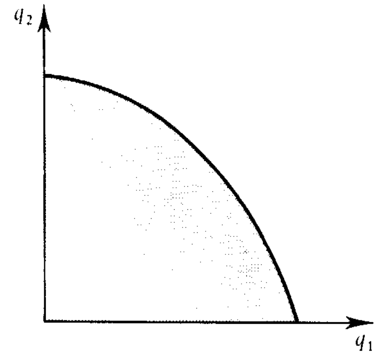
The Pareto set (endpoints excluded) must lie all above or all below or be coincident with the diagonal of the Edgeworth box. If it ever cuts the diagonal then because of constant returns, the isoquants of the two firms must in fact be tangent all along the diagonal, and so the diagonal must be the Pareto set (see also Exercise 15.B.7). Moreover, you should convince yourself of the correctness of the following claims.

Exercise 15.D.1: Suppose that the Pareto set of the 2 × 2 production model does not coincide with the diagonal of the Edgeworth box.

(a) Show that in this case, the factor intensity (the ratio of a firm's use of factor 1 relative to factor 2) of one of the firms exceeds that of the other at every point along the Pareto set.

(b) Show that in this case, any ray from the origin of either of the firms can intersect the Pareto set at most once. Conclude that the factor intensities of the two firms and the supporting relative factor prices change monotonically as we move along the Pareto set from one origin to the other.

In Figure 15.D.3, we depict the set of nonnegative output pairs (q_1, q_2) that can be produced using the economy's available factor inputs. This set is known as the *production possibility set*. Output pairs on the frontier of this set arise from factor allocations lying in the Pareto set of Figure 15.D.2(b). (Exercise 15.D.2 asks you to prove that the production possibility set is convex, as shown in Figure 15.D.3.)



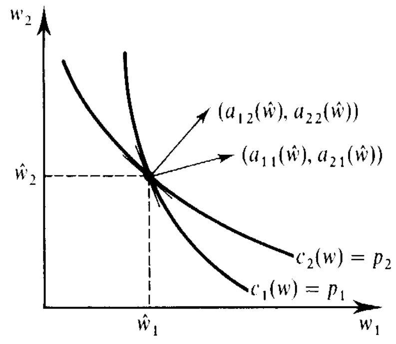


Figure 15.D.3 (left)
The production possibility set.

Figure 15.D.4 (right)
The equilibrium factor prices and factor intensities in an interior equilibrium.

With the purpose of examining more closely the determinants of the equilibrium factor allocation (z_1^*, z_2^*) and the corresponding equilibrium factor prices $w^* = (w_1^*, w_2^*)$, we now assume that the *factor intensities* of the two firms bear a systematic relation to one another. In particular, we assume that in the production of good 1, there is, relative to good 2, a greater need for the first factor. In Definition 15.D.1 we make precise the meaning of "greater need".

Definition 15.D.1: The production of good 1 is *relatively more intensive in factor 1* than is the production of good 2 if

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)}$$

at *all* factor prices $w = (w_1, w_2)$.

To determine the equilibrium factor prices, suppose that we have an *interior* equilibrium in which the production levels of the two goods are strictly positive (otherwise, we say that the equilibrium is *specialized*). Given our constant returns assumption, a necessary condition for (w_1^*, w_2^*) to be the factor prices in an interior equilibrium is that it satisfies the system of equations

$$c_1(w_1, w_2) = p_1 \quad \text{and} \quad c_2(w_1, w_2) = p_2. \tag{15.D.7}$$

That is, at an interior equilibrium, prices must be equal to unit cost. This gives us two equations for the two unknown factor prices w_1 and w_2.[13]

Figure 15.D.4 depicts the two unit cost functions in (15.D.7). By expression (15.D.7), a necessary condition for $(\hat{w}_1, \hat{w}_2)$ to be the factor prices of an interior equilibrium is that these curves cross at $(\hat{w}_1, \hat{w}_2)$. Moreover, the factor intensity assumption implies that whenever the two curves cross, the curve for firm 2 must be flatter (less negatively sloped) than that for firm 1 [recall that $\nabla c_j(w) = (a_{1j}(w), a_{2j}(w))$]. From this, it follows that the two curves can cross at most once.[14] Hence, under the

13. Expression (15.D.7) is the constant returns version of (15.D.3). Note that the effect of the constant returns to scale assumption is to make (15.D.3) independent of the output levels $(q_1, \ldots, q_J)$ (for interior equilibria).

14. If they crossed several times, then the curve for firm 2 must cross the curve for firm 1 at least once from above. At this crossing point, the curve for firm 2 would be steeper than the curve for firm 1, contradicting the factor intensity condition.

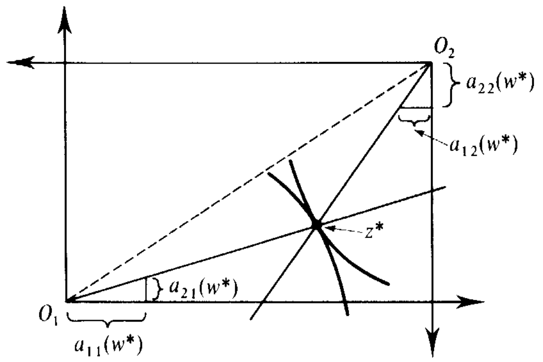


Figure 15.D.5
The equilibrium factor allocation.

factor intensity condition, there is at most a single pair of factor prices that can arise as the equilibrium factor prices of an interior equilibrium.[15]

Once the equilibrium factor prices w^* are known, the equilibrium output levels can be found graphically by determining the unique point (z_1^*, z_2^*) in the Edgeworth box of factor allocations at which both firms have the factor intensities associated with factor prices w^*, that is,

$$\frac{z_{11}^*}{z_{21}^*} = \frac{a_{11}(w^*)}{a_{21}(w^*)} \quad \text{and} \quad \frac{z_{12}^*}{z_{22}^*} = \frac{a_{12}(w^*)}{a_{22}(w^*)}.$$

The construction is depicted in Figure 15.D.5.

An important consequence of this discussion is that in the 2 × 2 production model, if the factor intensity condition holds, then as long as the economy does not specialize in the production of a single good [and therefore (15.D.7) holds], the equilibrium factor prices depend *only on the technologies of the two firms and on the output prices p*. Thus, the levels of the endowments matter only to the extent that they determine whether the economy specializes. This result is known in the international trade literature as the *factor price equalization theorem.* The theorem provides conditions (which include the presence of tradable consumption goods, identical production technologies in each country, and price-taking behavior) under which the prices of nontradable factors are equalized across nonspecialized countries.

We now present two comparative statics exercises. We first ask: How does a change in the price of one of the outputs, say p_1, affect the equilibrium factor prices and factor allocations? Figure 15.D.6(a), which depicts the induced change in Figure 15.D.4, identifies the change in factor prices. The increase in p_1 shifts firm 1's curve

15. Note, however, that although $(\hat{w}_1, \hat{w}_2)$ may solve (15.D.7), this is not sufficient to ensure that $(\hat{w}_1, \hat{w}_2)$ are equilibrium factor prices. In particular, even though $(\hat{w}_1, \hat{w}_2)$ solve (15.D.7), no interior equilibrium may exist. In Exercise 15.D.6, you are asked to show that under the factor intensity condition, the equilibrium will involve positive production of the two goods if and only if

$$\frac{a_{11}(\hat{w})}{a_{21}(\hat{w})} > \frac{\bar{z}_1}{\bar{z}_2} > \frac{a_{12}(\hat{w})}{a_{22}(\hat{w})},$$

where $\hat{w} = (\hat{w}_1, \hat{w}_2)$ is the unique solution to (15.D.7). In words, the factor intensity of the overall economy must be intermediate between the factor intensities of the two firms computed at the sole vector of factor prices at which diversification can conceivably occur.

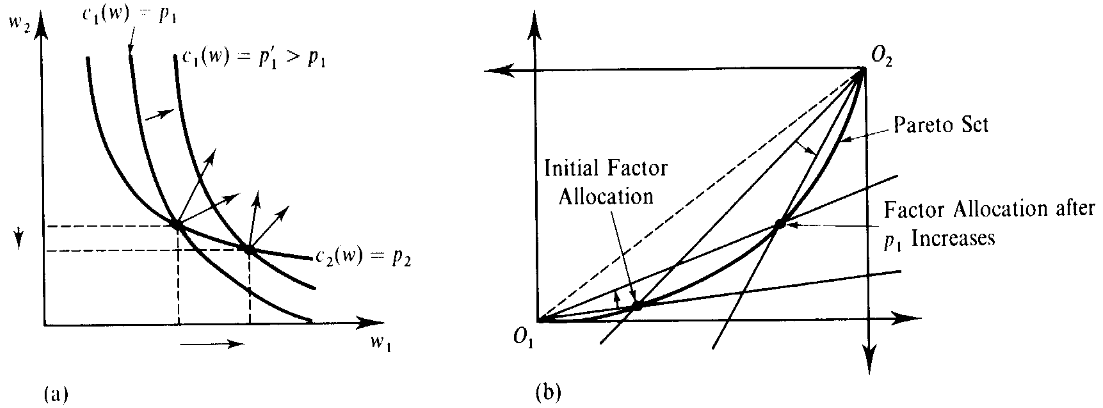


Figure 15.D.6 The Stolper–Samuelson theorem. (a) The change in equilibrium factor prices. (b) The change in the equilibrium factor allocation.

[the set $\{(w_1, w_2): c_1(w_1, w_2) = p_1\}$] outward toward higher factor price levels; the point of intersection of the two curves moves out along firm 2's curve to a higher level of w_1 and a lower level of w_2.

Formally, this gives us the result presented in Proposition 15.D.1.

Proposition 15.D.1: (*Stolper Samuelson Theorem*) In the 2 × 2 production model with the factor intensity assumption, if p_j increases, then the equilibrium price of the factor more intensively used in the production of good j increases, while the price of the other factor decreases (assuming interior equilibria both before and after the price change).[16]

Proof: For illustrative purposes, we provide a formal proof to go along with the graphical analysis of Figure 15.D.6 presented above. Note that it suffices to prove the result for an infinitesimal change $dp = (1, 0)$.

Differentiating conditions (15.D.7), we have

$$dp_1 = \nabla c_1(w^*)\cdot dw = a_{11}(w^*)\, dw_1 + a_{21}(w^*)\, dw_2,$$
$$dp_2 = \nabla c_2(w^*)\cdot dw = a_{12}(w^*)\, dw_1 + a_{22}(w^*)\, dw_2,$$

or in matrix notation,

$$dp = \begin{bmatrix} a_{11}(w^*) & a_{21}(w^*) \\ a_{12}(w^*) & a_{22}(w^*) \end{bmatrix} dw.$$

16. See Exercise 15.D.3 for a strengthening of this conclusion. We also note that, strictly speaking, the factor inensity condition is not required for this result. The reason is that, as we saw in Exercise 15.D.1, the firm that uses one factor, say factor 1, more intensely is the same for any point in the Pareto set of factor allocations. Suppose, for example, that we are as in Figure 15.D.2(b), where firm 1 uses factor 1 more intensively. Then, when p_1 rises, we can see from Figure 15.D.3, and the overall revenue-maximizing property of equilibrium discussed earlier in this section, that the output of good 1 increases and that of good 2 decreases. This implies that we move along the Pareto set in Figure 15.D.2(b) toward firm 2's origin. Therefore, recalling Exercise 15.D.1, both firms' intensity of use of factor 1 decreases. Hence, the equilibrium factor price ratio w_1^*/w_2^* must increase. Finally, since firm 2 is still breaking even and its output price has not changed, this implies that w_1^* increases and w_2^* decreases.

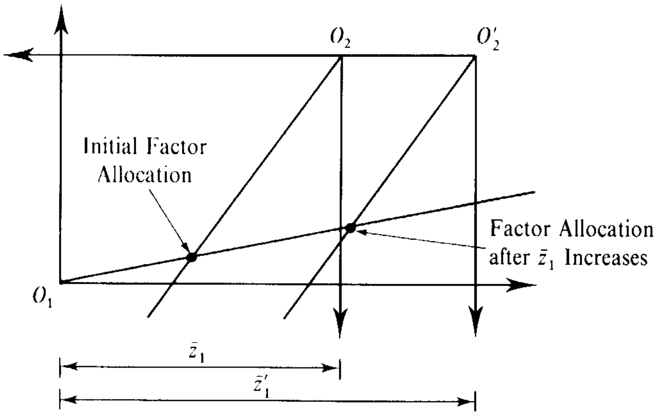


Figure 15.D.7 The Rybcszynski theorem.

Denote this 2×2 matrix by A. The factor intensity assumption implies that $|A| = a_{11}(w^*)a_{22}(w^*) - a_{12}(w^*)a_{21}(w^*) > 0$. Therefore A^{-1} exists and we can compute it to be

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22}(w^*) & -a_{21}(w^*) \\ -a_{12}(w^*) & a_{11}(w^*) \end{bmatrix}.$$

Hence, the entries of A^{-1} are positive at the diagonal and negative off the diagonal. Since $dw = A^{-1}\,dp$, this implies that for $dp = (1, 0)$ we have $dw_1 > 0$ and $dw_2 < 0$, as we wanted. ■

We have just seen that if p_1 increases, then w_1^*/w_2^* increases. Therefore, both firms must move to a less intensive use of factor 1. Figure 15.D.6(b) depicts the resulting change in the equilibrium allocation of factors. As can be seen, the factor allocation moves to a new point in the Pareto set at which the output of good 1 has risen and that of good 2 has fallen.

For the second comparative statics exercise, suppose that the total availability of factor 1 increases from $\bar{z}_1$ to $\bar{z}'_1$. What is the effect of this on equilibrium factor prices and output levels? Because neither the output prices nor the technologies have changed, the factor input prices remain unaltered (as long as the economy does not specialize). As a result, factor intensities also do not change. The new input allocation is then easily determined in the superimposed Edgeworth boxes of Figure 15.D.7; we merely find the new intersection of the two rays associated with the unaltered factor intensity levels.

Thus, examination of Figure 15.D.7 gives us the result presented in Proposition 15.D.2.

Proposition 15.D.2: (*Rybcszynski Theorem*) In the 2×2 production model with the factor intensity assumption, if the endowment of a factor increases, then the production of the good that uses this factor relatively more intensively increases and the production of the other good decreases (assuming interior equilibria both before and after the change of endowment).

For further discussion of the 2×2 production model see, for example, Johnson (1971).

Consider the general case of an arbitrary number of factors L and outputs J. For given output prices, the zero-profit conditions [i.e., the general analog of expression (15.D.7)] constitute a (nonlinear) system of J equations in L unknowns. If $L > J$, then there are too many unknowns and we cannot hope that the zero-profit conditions alone will determine the factor prices. The total factor endowments will play a role. If $J > L$, then there are too many equations and, for typical world prices, they cannot all be satisfied simultaneously. What this means is that the economy will specialize in the production of a number of goods equal to the number of factors L. The set of goods chosen may well depend on the endowments of factors. Beyond the 2×2 situation (the analysis of which, as we have seen, is quite instructive), the case $L = J$ seems too coincidental to be of interest. Nevertheless, we point out that in this case the zero-profit conditions are nonlinear and that in order to guarantee a unique solution (and versions of the Stolper–Samuelson and the Rybcszynski theorems), we need a generalization of the factor intensity condition. These generalizations exist, but they cannot be interpreted economically in as simple a manner as can the factor intensity condition of the 2×2 model.

15.E General Versus Partial Equilibrium Theory

There are some problems that are inherently general equilibrium problems. It would be hard to envision convincing analyses of economic growth, demographic change, international economic relations, or monetary policy that were restricted to only a subset of commodities and did not consider economy-wide feedback effects.

Partial equilibrium models of markets, or of systems of related markets, determine prices, profits, productions, and the other variables of interest adhering to the assumption that there are no feedback effects from these endogenous magnitudes to the underlying demand or cost curves that are specified in advance. Individuals' wealth is another variable that general equilibrium theory regards as endogenously determined but that is often treated as exogenous in partial equilibrium theory.

If general equilibrium analysis did not change any of the predictions or conclusions of partial equilibrium analysis, it would be of limited significance when applied to problems amenable to partial equilibrium treatment. It might be of comfort because we would then know that our partial equilibrium conclusions are valid, but it would not change our view of how markets work. However, things are not that simple. The choice of methodology may be far from innocuous. We now present an example [due to Bradford (1978)] in which a naive application of partial equilibrium analysis leads us seriously astray. See Sections 3.I and 10.G for some discussion of when partial equilibrium theory is (approximately) justified.

A Tax Incidence Example

Consider an economy with a large number of towns, N. Each town has a single price-taking firm that produces a consumption good by means of the strictly concave production function $f(z)$ (once again, we could reinterpret the model as having many identical firms in each town to make the price-taking hypothesis more palatable). This consumption good, which is identical across towns, is traded in a national market. The overall economy has M units of labor, inelastically supplied by workers who derive utility only from the output of the firms. Workers are free to move from town to town and do so to seek the highest wage. We normalize the price of the consumption good to be 1, and we denote the wage rate in town n's labor market by w_n.

Given that workers can move freely in search of the highest wage, at an equilibrium the wage rates across towns must be equal; that is, we must have $w_1 = \cdots = w_N = \bar{w}$. From the symmetry of the problem, it must be that each firm hires exactly M/N units of labor in an equilibrium. As a result, the equilibrium wage rate must be $\bar{w} = f'(M/N)$. The equilibrium profits of an individual firm are therefore $f(M/N) - f'(M/N)(M/N)$.

Now suppose that town 1 levies a tax on the labor used by the firm located there. We investigate the "incidence" of the tax on workers and firms (or, more properly, on the firms' owners); that is, we examine the extent to which each group bears the burden of the tax. If the tax rate is t and the wage in town 1 is w_1, the labor demand of the firm in town 1 will be the amount z_1 such that $f'(z_1) = t + w_1$. At this point, we may be tempted to argue that, since N is large, we can approximate and take the wage rates elsewhere, $\bar{w}$, to be unaffected by this change in town 1. Moreover, since labor moves freely, the supply correspondence of workers in town 1 should then be 0 at $w_1 < \bar{w}$, ∞ at $w_1 > \bar{w}$, and $[0, \infty]$ at $w_1 = \bar{w}$. Thus, taking a partial equilibrium view, the equilibrium wage rate in the town 1 labor market remains equal to $\bar{w}$, and the labor employed in town 1 falls to the level z_1 such that $f'(z_1) = t + \bar{w}$ (hence, some labor will shift to the other towns). By adopting this sort of partial equilibrium view of the labor market of town 1, we are therefore led to conclude that the income of workers remains the same, as does the profit of every firm not located in town 1. Only the profit of the firm in town 1 decreases. The qualitative conclusion is that firms (actually, firms' owners) "bear" all of the tax burden. Labor, because it is free to move and because the number of untaxed firms is large, "escapes."

Alas, this conclusion constitutes an egregious mistake, and it will be overturned by a general equilibrium view of the same model.

We now look at the general equilibrium across the labor markets of all the towns. We know that the equilibrium wage rate must be such that $w_1 = \cdots = w_N$ and that all M units of labor are employed. Let $w(t)$ be this common equilibrium wage when the tax rate in town 1 is t. By symmetry, the firms in towns $2, \ldots, N$ will each employ the same amount of labor, $z(t)$. Let $z_1(t)$ be the equilibrium labor demand of the firm in town 1 when town 1's tax rate is t. Then the equilibrium conditions are

$$(N-1)z(t) + z_1(t) = M. \tag{15.E.1}$$

$$f'(z(t)) = w(t). \tag{15.E.2}$$

$$f'(z_1(t)) = w(t) + t. \tag{15.E.3}$$

Consider the impact on wages of the introduction of a small tax dt. Substituting from (15.E.1) for $z_1(t)$ in (15.E.3), differentiating with respect to t, and evaluating at $t = 0$ [at which point $z_1(0) = z(0) = (M/N)$], we get

$$-f''(M/N)(N-1)z'(0) = w'(0) + 1. \tag{15.E.4}$$

But from (15.E.2), we get

$$f''(M/N)z'(0) = w'(0). \tag{15.E.5}$$

Substituting from (15.E.5) into (15.E.4) yields

$$w'(0) = -\frac{1}{N}.$$

Therefore, once the general equilibrium effects are taken into account, we see that the wage rate in all towns falls with the imposition of the tax in town 1. However, we see that this fall in the wage rate approaches zero as N grows large. Thus, at this point, it may still seem that our partial equilibrium approximation will have given us the correct answers for large N. But this is not so. Consider the effect of the tax on total profits. The partial equilibrium approach told us that workers escaped the tax; all the tax fell as a burden on firms. But letting $\pi(w)$ be the profit function of a representative firm, the change in aggregate profits from the imposition of this tax is[17]

$$(N-1)\pi'(\bar{w})w'(0) + \pi'(\bar{w})(w'(0)+1) = \pi'(\bar{w})\left(-\frac{N-1}{N} + \frac{N-1}{N}\right) = 0.$$

Aggregate profits stay constant! Thus, all of the burden of a small tax falls on laborers, not on the owners of firms. Although the partial equilibrium approximation is correct as far as getting prices and wages about right, it errs by just enough, and in just such a direction, that the conclusions of the tax incidence analysis based on it are completely reversed.[18]

17. Recall that the profits of the firm in town 1 are $\pi(w(t)+t)$.

18. We note that the justifications of partial equilibrium analysis in terms of small individual budget shares that we informally described in Sections 3.I and 10.G do not apply here because the "consumption" goods in this example (jobs in different towns) are perfect substitutes and therefore individual budget shares are not guaranteed to be small at all prices.

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EXERCISES

15.B.1[A] Consider an Edgeworth box economy in which the two consumers have locally nonsatiated preferences. Let $x_{\ell i}(p)$ be consumer i's demand for good ℓ at prices $p = (p_1, p_2)$.

(a) Show that $p_1(\sum_i x_{1i}(p) - \bar{\omega}_1) + p_2(\sum_i x_{2i}(p) - \bar{\omega}_2) = 0$ for all prices p.

(b) Argue that if the market for good 1 clears at prices $p^* \gg 0$, then so does the market for good 2; hence, p^* is a Walrasian equilibrium price vector.

15.B.2[A] Consider an Edgeworth box economy in which the consumers have the Cobb–Douglas utility functions $u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha}$ and $u_2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta}$. Consumer i's endowments are $(\omega_{1i}, \omega_{2i}) \gg 0$, for $i = 1, 2$. Solve for the equilibrium price ratio and allocation. How do these change with a differential change in ω_{11}?

15.B.3[B] Argue (graphically) that in an Edgeworth box economy with locally nonsatiated preferences, a Walrasian equilibrium is Pareto optimal.

15.B.4[C] Consider an Edgeworth box economy. An offer curve has the *gross substitute* property if an increase in the price of one commodity decreases the demand for that commodity and increases the demand for the other one.

(a) Represent in an Edgeworth box the shape of an offer curve with the gross substitute property.

(b) Assume that the offer curves of the two consumers have the gross substitute property. Show then that the offer curves can intersect only once (not counting the intersection at the initial endowments).

Let us denote an offer curve as *normal* if an increase in the price of one commodity leads to an increase in the demand for that commodity only if the demands of the two commodities both increase.

(c) Represent in the Edgeworth box the shape of a normal offer curve that does not satisfy the gross substitute property.

(d) Show that there are preferences giving rise to offer curves that are not normal. Show that the demand function for such preferences is not normal (i.e., at some prices some good is inferior).

(e) Show in the Edgeworth box that if the offer curve of one consumer is normal and that of the other satisfies the gross substitute property, then the offer curves can intersect at most once (not counting the intersection at the initial endowments).

(f) Show that two normal offer curves can intersect several times.

15.B.5[A] Verify that the offer curves of Example 15.B.2 are as claimed. Solve also for the claimed values of relative prices.

15.B.6[B] (D. Blair) Compute the equilibria of the following Edgeworth box economy (there is more than one):

$$u_1(x_{11}, x_{21}) = (x_{11}^{-2} + (12/37)^3 x_{21}^{-2})^{-1/2}, \qquad \omega_1 = (1, 0),$$
$$u_2(x_{12}, x_{22}) = ((12/37)^3 x_{12}^{-2} + x_{22}^{-2})^{-1/2}, \qquad \omega_2 = (0, 1).$$

15.B.7[C] Show that if both consumers in an Edgeworth box economy have continuous, strongly monotone, and strictly convex preferences, then the Pareto set has no "holes": precisely, it is a connected set. Show that if, in addition, the preferences of both consumers are homothetic, then the Pareto set lies entirely on one side of the diagonal of the box.

15.B.8[B] Suppose that both consumers in an Edgeworth box have continuous and strictly convex preferences that admit a quasilinear utility representation with the first good as numeraire. Show that any two Pareto optimal allocations in the interior of the Edgeworth box then involve the same consumptions of the second good. Connect this with the discussion of Chapter 10.

15.B.9[B] Suppose that in a pure exchange economy (i.e., an economy without production), we have two consumers, Alphanse and Betatrix, and two goods, Perrier and Brie. Alphanse and Betatrix have the utility functions:

$$u_\alpha = \text{Min}\{x_{p\alpha}, x_{b\alpha}\} \qquad \text{and} \qquad u_\beta = \text{Min}\{x_{p\beta}, (x_{b\beta})^{1/2}\}$$

(where $x_{p\alpha}$ is Alphanse's consumption of Perrier, and so on). Alphanse starts with an endowment of 30 units of Perrier (and none of Brie); Betatrix starts with 20 units of Brie (and none of Perrier). Neither can consume negative amounts of a good. If the two consumers behave as price takers, what is the equilibrium?

Suppose instead that Alphanse begins with only 5 units of Perrier while Betatrix's initial endowment remains 20 units of Brie, 0 units of Perrier. What happens now?

15.B.10[C] (*The Transfer Paradox*) In a two-consumer, two-commodity pure exchange economy with continuous, strictly convex and strongly monotone preferences, consider the comparative statics of the welfare of consumer 1 with changes in the initial endowments $\omega_1 = (\omega_{11}, \omega_{21})$ and $\omega_2 = (\omega_{12}, \omega_{22})$.

(a) Suppose first that the preferences of the two consumers are quasilinear with respect to the same numeraire. Show that if the endowments of consumer 1 are increased to $\omega'_1 \gg \omega_1$ while ω_2 remains the same, then at equilibrium the utility of consumer 1 *may* decrease. Interpret this observation and relate it to the theory of a quantity-setting monopoly.

(b) Suppose now that the increase in resources of consumer 1 constitute a transfer from consumer 2, that is, $\omega'_1 = \omega_1 + z$ and $\omega'_2 = \omega_2 - z$ with $z \geq 0$. Under the same assumption as in **(a)**, show that the utility of consumer 1 cannot decrease.

(c) Consider again a transfer as in **(b)**, but this time preferences may not be quasilinear. Suppose that the transfer z is small and that similarly the change in the equilibrium (relative) price is restricted to be small. Show that it is possible for the utility of consumer 1 to decrease (this is called the *transfer paradox*). A graphical illustration in the Edgeworth box suffices to make the point. Interpret in terms of the interplay between substitution and wealth effects.

(d) Show that in this Edgeworth box example (but, be warned, not more generally) the transfer paradox can happen only if there is a multiplicity of equilibria. [*Hint:* Argue graphically in the Edgeworth box. Show that if a transfer to consumer 1 leads to a decrease of the utility of consumer 1, then there must be an equilibrium at the no-transfer situation where consumer 1 gets an even lower level of utility.]

15.C.1[B] This exercise refers to the one-consumer, one-firm economy discussed in Section 15.C.

(a) Prove that in an economy with one firm, one consumer, and strictly convex preferences and technology, the equilibrium level of production is unique.

(b) Fix the price of output to be 1. Define the excess demand function for labor as

$$z_1(w) = x_1(w, w\bar{L} + \pi(w)) + y_1(w) - \bar{L},$$

where w is the wage rate, $\pi(\cdot)$ is the profit function, and $x_1(\cdot, \cdot)$, $y_1(\cdot)$ are, respectively, the consumer's demand function for leisure and the firm's demand function for labor. Show that the slope of the excess demand function is not necessarily of one sign throughout the range of prices but that it is necessarily negative in a neighborhood of the equilibrium.

(c) Give an example to show that there can be multiple equilibria in a strictly convex economy with one firm and two individuals, each of whom is endowed with labor alone. (Assume that profits are split equally between the two consumers.) Can this happen if the firm operates under constant rather than strictly decreasing returns to scale?

15.C.2[A] Consider the one-consumer, one-producer economy discussed in Section 15.C. Compute the equilibrium prices, profits, and consumptions when the production function is $f(z) = z^{1/2}$, the utility function is $u(x_1, x_2) = \ln x_1 + \ln x_2$, and the total endowment of labor is $\bar{L} = 1$.

15.D.1[B] In text.

15.D.2[A] Show that in the 2×2 production model the production possibility set is convex (assume free disposal).

15.D.3[B] Show that the Stolper–Samuelson theorem (Proposition 15.D.1) can be strengthened to assert that the increase in the price of the intensive factor is proportionally larger than the increase in the price of the good (and therefore the well-being of a consumer who owns only the intensive factor must increase).

15.D.4[C] Consider a general equilibrium problem with two consumer–workers ($i = 1, 2$), two constant returns firms ($j = 1, 2$) with concave technologies, two factors of production ($\ell = 1, 2$), and two consumption goods ($j = 1, 2$) produced, respectively, by the two firms. Assume that the production of consumption good 1 is relatively more intensive in factor 1. Neither consumer consumes either of the factors. Consumer 1 owns one unit of factor 1 while consumer 2 owns one unit of factor 2.

(a) Set up the equilibrium problem as one of clearing the factor and goods markets (in a closed economy context) under the assumption that prices are taken as given and productions are profit maximizing.

(b) Suppose that consumer 1 has a taste only for the second consumption good and that consumer 2 cares only for the first good. Argue that there is at most one equilibrium.

(c) Suppose now that consumer 1 has a taste only for the first good and that consumer 2 cares only for the second good. Argue that a multiplicity of equilibria is possible.

[*Hint:* Parts **(b)** and **(c)** can be answered by graphical analysis in the Edgeworth box of factors of production.]

15.D.5[B] Show that the Rybcszynski theorem (Proposition 15.D.2) can be strengthened to assert that the proportional increase in the production of the good that uses the increased factor relatively more intensively is greater than the proportional increase in the endowment of the factor.

15.D.6[C] Suppose you are in the 2×2 production model with output prices (p_1, p_2) given (the economy could be a small open economy). The factor intensity condition is satisfied (production of consumption good 1 uses factor 1 more intensely). The total endowment vector is $\bar{z} \in \mathbb{R}^2$.

(a) Set up the equilibrium conditions for factor prices (w_1^*, w_2^*) and outputs (q_1^*, q_2^*) allowing for the possibility of specialization.

(b) Suppose that $\hat{w} = (\hat{w}_1, \hat{w}_2)$ are factor prices with the property that for each of the two goods the unit cost equals the price. Show that the necessary and sufficient condition for the equilibrium determined in **(a)** to have $(q_1^*, q_2^*) \gg 0$ is that $\bar{z}$ belongs to the set

$$\{(z_1, z_2) \in \mathbb{R}^2_+ : a_{11}(\hat{w})/a_{21}(\hat{w}) > z_1/z_2 > a_{12}(\hat{w})/a_{22}(\hat{w})\},$$

where $a_{\ell j}(\hat{w})$ is the optimal usage (at factor prices $\hat{w}$) of the input ℓ in the production of one unit of good j. This set is called the *diversification cone*.

(c) The unit-dollar isoquant of good j is the set of factor combinations that produce an amount of good j of 1 dollar value. Show that under the factor intensity condition the unit-dollar isoquants of the two goods can intersect at most once. Use the unit-dollar isoquants to construct graphically the diversification cone. [*Hint:* If they intersect twice then there are two points (one in each isoquant) proportional to each other and such that the slopes of the isoquants at these points are identical.]

(d) When the total factor endowment is not in the diversification cone, the equilibrium is specialized. Can you determine, as a function of total factor endowments, in which good the economy will specialize and what the factor prices will be? Be sure to verify the inequality conditions in **(a)**. To answer this question you can make use of the graphical apparatus developed in **(c)**.

15.D.7$^{\mathrm{B}}$ Suppose there are two output goods and two factors. The production functions for the two outputs are

$$f_1(z_{11}, z_{21}) = 2(z_{11})^{1/2} + (z_{21})^{1/2} \quad \text{and} \quad f_2(z_{12}, z_{22}) = (z_{12})^{1/2} + 2(z_{22})^{1/2}.$$

The international prices for these goods are $p = (1, 1)$. Firms are price takers and maximize profits. The total factor endowments are $\bar{z} = (\bar{z}_1, \bar{z}_2)$. Consumers have no taste for the consumption of factors of production. Derive the equilibrium factor allocation $((z_{11}^*, z_{21}^*), (z_{12}^*, z_{22}^*))$ and the equilibrium factor prices (w_1^*, w_2^*) as a function of $(\bar{z}_1, \bar{z}_2)$. Verify that you get the same result whether you proceed through equations (15.D.1) and (15.D.2) or by solving (15.D.5).

15.D.8$^{\mathrm{B}}$ The setting is as in the 2×2 production model. The production functions for the two outputs are of the Cobb–Douglas type:

$$f_1(z_{11}, z_{21}) = (z_{11})^{2/3}(z_{21})^{1/3} \quad \text{and} \quad f_2(z_{12}, z_{22}) = (z_{12})^{1/3}(z_{22})^{2/3}.$$

The international output price vector is $p = (1, 1)$ and the total factor endowments vector is $\bar{z} = (\bar{z}_1, \bar{z}_2) \gg 0$. Compute the equilibrium factor allocations and factor prices for all possible values of $\bar{z}$. Be careful in specifying the region of total endowment vectors where the economy will specialize in the production of a single good.

15.D.9$^{\mathrm{C}}$ (*The Heckscher–Ohlin Theorem*) Suppose there are two consumption goods, two factors, and two countries A and B. Each country has technologies as in the 2×2 production model. The technologies for the production of each consumption good are the same in the two countries. The technology for the production of the first consumption good is relatively more intensive in factor 1. The endowments of the two factors are $\bar{z}_A \in \mathbb{R}^2_+$ and $\bar{z}_B \in \mathbb{R}^2_+$ for countries A and B, respectively. We assume that country A is relatively better endowed with factor 1, that is, $\bar{z}_{1A}/\bar{z}_{2A} > \bar{z}_{1B}/\bar{z}_{2B}$. Consumers are identical within and between countries. Their preferences are representable by increasing, concave, and homogeneous utility functions that depend only on the amount consumed of the two consumption goods.

Suppose that factors are not mobile and that each country is a price taker with respect to the international prices for consumption goods. Suppose then that at the international prices $p = (p_1, p_2)$ we have that, first, neither of the two countries specializes and, second, the international markets for consumption goods clear. Prove that country A must be exporting good 1, the good whose production is relatively more intensive in the factor that is relatively more abundant in country A.