CHAPTER

20

Equilibrium and Time

20.A Introduction

In this chapter, we present the basic elements of the extension of competitive equilibrium theory to an intertemporal setting. In the presentation, we try to maintain a balance between two possible approaches to the theory varying by the degree of emphasis on time.

A first approach contemplates equilibrium in time merely as the particular case of the general theory developed in the previous chapters in which commodities are indexed by time as one of the many characteristics defining them. This is a useful point of view (the display of the underlying unity of seemingly disparate phenomena is one of the prime roles of theory), and to a point we build on it. However, exclusive reliance on this approach would, in the limit, be self-defeating. It would reduce this chapter to a footnote to the preceding ones.

A second approach proceeds by stressing, rather than deemphasizing, the special structure of time. Again, we follow this line to some degree. Thus, every model discussed in this chapter accepts the open-ended infinity of time, or the fact that production takes time. Also, at the cost of some generality, we pursue our treatment under assumptions of stationarity and time separability that allow for a sharp presentation of the dynamic aspects of the theory.

Sections 20.B and 20.C are concerned with the description of, respectively, the consumption and the production sides of the economy.

Section 20.D is the heart of this chapter. It deals with the basic properties of equilibria (including definitions, existence, optimality, and computability) in the context of a single-consumer economy.

Section 20.E (which concentrates on steady states) and Section 20.F (which is general) study the dynamics of the single-consumer model.

Section 20.G considers economies with several consumers. The message of this section is that, as long as the Pareto optimality of equilibrium is guaranteed, the qualitative aspects of the positive theory of Chapter 17 extend to the more general situation and, moreover, that the properties of individual equilibria identified by the single-consumer methodology remain valid in the broader context.

Section 20.H gives an extremely succinct account of overlapping-generations economies, a model of central importance in modern macroeconomic theory. Our interest in it is twofold: on the one hand, we want to display it as yet another instance of a useful equilibrium model; on the other hand, we want to point out that it is an example that, because of the infinity of generations, does not fit the general model of Section 20.G, and one that gives rise to some new and interesting issues having to do with the optimality and the multiplicity of equilibria.

Section 20.I gathers some remarks on nonequilibrium considerations (short-run equilibrium and tâtonnement stability, learning, and so on).

For pedagogical purposes, the entire chapter deals only with the deterministic version of the theory. The unfolding of time is a line, not a tree. A full synthesis of the approaches of Chapter 19 (on uncertainty) and the current one (on time) is possible. However, we view its presentation as advanced material beyond the scope of this textbook. The account of Stokey and Lucas with Prescott (1989) constitutes an excellent introduction to the general theory.

A point of notation: in this chapter $\sum_t$ always means $\sum_{t=0}^{t=\infty}$, that is, $\lim_{T\to\infty}\sum_{t=0}^{t=T}$. When the sum does not run from $t = 0$ to $t = \infty$ the two end-points of the sum are explicitly indicated.

20.B Intertemporal Utility

In this chapter, we assume that there are infinitely many dates $t = 0, 1, \ldots,$ and that the objects of choice for consumers are *consumption streams* $c = (c_0, \ldots, c_t, \ldots)$ where $c_t \in \mathbb{R}^L_+$, $c_t \geq 0$.[1] To keep things simple, we will consider only consumption streams that are *bounded*, that is, that have $\sup_t \|c_t\| < \infty$.

Rather than proceed from the most general form of preferences over consumption streams to the more specific, we instead introduce first the very special form that we assume throughout this chapter (except for Sections 20.H and 20.I); we subsequently discuss its special properties from a general point of view.

It is customary in intertemporal economies to assume that preferences over consumption streams $c = (c_0, \ldots, c_t, \ldots)$ can be represented by a utility function $V(c)$ having the special form

$$V(c) = \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{20.B.1}$$

where $\delta < 1$ is a *discount* factor and $u(\cdot)$, which is defined on $\mathbb{R}^L_+$, is strictly increasing and concave. This chapter will be no exception to this rule: Throughout it we assume that preferences over consumption streams take this form. However, we comment here, at some length, on six aspects of this utility function. As a matter of notation, given a consumption stream $c = (c_0, \ldots, c_t, \ldots)$, we let $c^T = (c_0^T, c_1^T, \ldots)$ denote the T-period "backward shift" consumption stream, that is, the stream $(c_0^T, c_1^T, \ldots)$ with $c_t^T = c_{t+T}$ for all $t \geq 0$.

(1) *Time impatience.* The requirement that future utility is discounted (i.e., that $\delta < 1$), implies *time impatience*. That is, if $c = (c_0, c_1, \ldots, c_t, \ldots)$ is a nonzero consumption stream, then the (forward-) shifted consumption stream $c' = (0, c_0, c_1, \ldots, c_{t-1}, \ldots)$ is strictly worse than c (see Exercise 20.B.1). It is an

1. We use the terms "stream," "trajectory," "program," and "path" synonymously.

assumption that is very helpful in guaranteeing that a bounded consumption stream has a finite utility value [i.e., guarantees that the sum in (20.B.1) converges], thus allowing us to compare any two such consumption streams[2] and making possible the application of the machinery of the calculus. There is a strand of opinion that views this technical convenience as the real reason for the fundamental role that the assumption of time discounting plays in economics. This skeptical view on the existence of substantive reasons[3] is excessive. An implication of time discounting is that the distant future does not matter much for current decisions, and this feature seems more realistic than its opposite.

A possible interpretation, and defense, of the discount factor δ views it as a probability of survival to the next period. Then $V(c)$ is the expected value of lifetime utility. For another interpretation, see (6) below.

(2) *Stationarity.* A more general form of the utility function would be

$$V(c) = \sum_{t=0}^{\infty} u_t(c_t). \tag{20.B.2}$$

The form (20.B.1) corresponds to the special case of (20.B.2) in which $u_t(c_t) = \delta^t u(c_t)$. This special form can be characterized in terms of *stationarity*. Consider two consumption streams $c \neq c'$ such that $c_t = c'_t$ for $t \leq T - 1$: that is, the two streams c and c' are one and the same up to period $T - 1$ and differ only after $T - 1$. Observe that the problem of choosing at $t = T$ between the current and future consumptions in c and c' is the same problem that a consumer would face at $t = 0$ in choosing between the consumption streams c^T and c'^T, the T backward shifts of c and c', respectively. Then *stationarity* requires that

$$V(c) \geq V(c') \quad \text{if and only if} \quad V(c^T) \geq V(c'^T).$$

It is a good exercise to verify that (20.B.1) satisfies the stationarity property and that the property can be violated by utility functions of the form $V(c) = \sum_t \delta_t^t u(c_t)$, that is, with a time-dependent discount factor (Exercise 20.B.2).

The property of stationarity should *not* be confused with the statement asserting that if the consumption streams c and c' coincide in the first $T - 1$ periods and a consumer chooses one of these streams at $t = 0$, then she will not change her mind at T. This "property" is tautologically true: at both dates we are comparing $V(c)$ and $V(c')$.[4] The stationarity experiment compares $V(c)$ and $V(c')$ at $t = 0$, but at period T it compares the utility values of the future streams shifted to $t = 0$, that is, $V(c^T)$ and $V(c'^T)$. Thus, stationarity says that in the context of the form (20.B.2), the preferences over the future are independent of the *age* of the decision maker.

Time stationarity is not essential to the analysis of this chapter (except for Sections 20.E and 20.F on dynamics), but it saves substantially on the use of subindices.

2. Hence, the completeness of the preference relation on consumption streams is guaranteed.

3. Ramsey (1928) called the assumption a "weakness of the imagination."

4. This property is often called *time consistency*. Time inconsistency is possible if tastes change through time (recall the example of Ulysses and the Sirens in Section 1.B!), but, as we have just argued, it must necessarily hold if the preference ordering over consumption streams $(c_0, \ldots, c_t, \ldots)$ does not change as time passes. In line with the entire treatment of Part IV, we maintain the assumption of unchanging tastes throughout the chapter.

(3) *Additive separability.* Two implications of the additive form of the utility function are that at any date T we have, first, that the induced ordering on consumption streams that begin at $T + 1$ is independent of the consumption stream followed from 0 to T, and, second, that the ordering on consumption streams from 0 to T is independent of whatever (fixed) consumption expectation we may have from $T + 1$ onward (see Exercise 20.B.3). In turn, these two separability properties imply additivity; that is, if the preference ordering over consumption streams satisfies these separability properties, then it can be represented by a utility function of the form $V(c) = \sum_t u_t(c_t)$ [this is not easy to prove, see Blackorby, Primont and Russell (1978)].

How restrictive is the assumption of additive separability? We can make two arguments in its favor: the first is technical convenience; the second is a vague sense that what happens far in the future or in the past should be irrelevant to the relative welfare appreciation of current consumption alternatives. Against it we have obvious counter-examples: Past consumption creates habits and addictions, the appreciation of a particularly wonderful dish may depend on how many times it has been consumed in the last week, and so on. There is, however, a very natural way to accommodate these phenomena within an additively separable framework. We could, for example, allow for the form $V(c) = \sum_t u_t(c_{t-1}, c_t)$. Here the utility at period t depends not only on consumption at date t but also on consumption at date $t - 1$ (or, more generally, on consumption at several past dates). We can formulate this in a slightly different way. Define a vector z_t of "habit" variables and a *household production technology* that uses an input vector c_{t-1} at $t - 1$ to jointly produce an output vector c_{t-1} of consumption goods at $t - 1$ and a vector $z_t = c_{t-1}$ of "habit" variables at t. Then, formally, u_t depends only on time t variables and total utility is $\sum_t u_t(z_t, c_t)$. In summary: additive separability is less restrictive than it appears if we allow for household production and a suitable number (typically larger than 1) of current variables.

(4) *Length of period.* The plausibility of the separability assumption, which makes the enjoyment of current consumption independent of the consumption in other periods, depends on the length of the period. Because even the most perishable consumption goods have elements of durability in them (in the form, for example, of a flow of "services" after the act of consumption), the assumption is quite strained if the length of the elementary period is very short. What determines the length of the period? To the extent that our model is geared to competitive theory, this period is institutionally determined: it should be an interval of time for which prices can be taken as constant. On a related point, note that the value of δ also depends, implicitly, on the length of the period. The shorter the period, the closer δ should be to 1.

(5) *Recursive utility.* With the form (20.B.1) for the utility function, we have $V(c) = u(c_0) + \delta V(c^1)$ for any consumption stream $c = (c_0, c_1, \ldots, c_t, \ldots)$. If we think of $u = u(c_0)$ as current utility and of $V = V(c^1)$ as future utility, we see that the marginal rate of substitution of current for future utility equals δ and is therefore independent of the levels of current and future utility. The *recursive utility model* [due to Koopmans (1960)] is a useful generalization of (20.B.1) that combines two features: it allows this rate to be variable but, as in the additively separable case, it has the property that the ordering of future consumption streams is independent of the consumption stream followed in the past.

The recursive model goes as follows. Denote current utility by $u \geq 0$ and future utility by $V \geq 0$. Then we are given a current utility function $u(c_t)$ and an *aggregator* function $G(u, V)$ that combines current and future utility into overall utility. For example, in the separable additive case we have $G(u, V) = u + \delta V$. More generally we could also have, for example, $G(u, V) = u^\alpha + \delta V^\alpha$, $0 < \alpha \leq 1$. In this case, the indifference curves in the (u, V) plane are not straight lines. The utility of a consumption stream $c = (c_0, \ldots, c_t, \ldots)$ could then be computed recursively from

$$V(c) = G(u(c_0), V(c^1)) = G(u(c_0), G(u(c_1), V(c^2))) = \cdots. \qquad (20.B.3)$$

For (20.B.3) to make sense we must be able to argue that the influence of $V(c^T)$ on $V(c)$ will become negligible as $T \to \infty$ [so that $V(c)$ can be approximately determined by taking a large T and letting $V(c^T)$ have an arbitrary value]. This amounts to an assumption of time impatience. In applications, it will typically not be necessary to compute $V(c)$ explicitly. See Exercise 20.B.4 for more on recursive utility.

(6) *Altruism.* The expression $V(c) = u(c_0) + \delta V(c^1)$ suggests a multigeneration interpretation of the single-consumer problem (20.B.1). Indeed, if generations live a single period and we think of generation 0 as enjoying her consumption according to $u(c_0)$, but caring also about the *utility* $V(c^1)$ of the next generation according to $\delta V(c^1)$, then $V(c) = u(c_0) + \delta V(c^1)$ is her overall utility. If every generation is similarly altruistic, then we conclude, by recursive substitution, that the objective function of generation 0 is precisely (20.B.1). The entire "dynasty" behaves as a single individual. With this we also have another justification for $\delta < 1$. The inequality means then that the members of the current generation care for their children, but not quite as much as for themselves. See Barro (1989) for more on these points.

20.C Intertemporal Production and Efficiency

Assume that there is an infinite sequence of dates $t = 0, 1, \ldots$. In each period t, there are L commodities. If it facilitates reading, you can take $L = 2$ and interpret the commodities as labor services and a generalized consumption–investment good (see Example 20.C.1). One of the great advantages of vector notation, however, is that in some cases—and this is one—there is no novelty involved in the general case. Thus, while you think you are understanding the simple problem, you are at the same time understanding the most general one.

We shall adopt the convention that goods are *nondurables.* This is a convention because, in order to make a good durable, it suffices to specify a storage technology whose role is, so to speak, to transport the commodity through time.

If we were exogenously endowed with some amount of resources (e.g., some initial capital and some amount of labor every period), we would ask what we could do with them. To give an answer, we need to specify the *production technology.* We already know from Chapter 5 how to do this formally by means of the concept of a *production set* (or a production transformation function, or a production function). With minimal loss of generality, we will restrict our technologies to be of the following form: the production possibilities at time t are entirely determined by the production decisions at the most recent past, that is, at time $t - 1$. If we keep in mind that we can always define new intermediate goods (such as different vintages of a machine), and also that we can always define periods to be very long, we see that the restriction is minor.

Thus, the technological possibilities at t will be formally specified by a production set $Y \subset \mathbb{R}^{2L}$ whose generic entries, or *production plans*, are written $y = (y_b, y_a)$. The indices b and a are mnemonic for "before" and "after." The interpretation is that the production plans in Y cover two periods (the "initial" and the "last" period) with $y_b \in \mathbb{R}^L$ and $y_a \in \mathbb{R}^L$ being, respectively, the production plans for the initial and the last periods. As usual, negative entries represent inputs and positive entries represent outputs.

We impose some assumptions on Y that are familiar from Section 5.B:

(i) Y is *closed and convex*.
(ii) $Y \cap \mathbb{R}^{2L}_+ = \{0\}$ (*no free lunch*).
(iii) $Y - \mathbb{R}^{2L}_+ \subset Y$ (*free disposal*).

An assumption specific to the temporal setting is the requirement that inputs not be used *later* than outputs are produced (i.e., production takes time). This is captured by

(iv) If $y = (y_b, y_a) \in Y$ then $(y_b, 0) \in Y$ (*possibility of truncation*).

In words, (iv) says that, whatever the production plans for the initial period, not producing in the last period is a possibility. A simple case is when $y_{at} \geq 0$ for every $y \in Y$, that is, when all inputs are used in the initial period. Then (iv) is implied by the free-disposal property (iii).

Example 20.C.1: *Ramsey–Solow Model.*[5] Assume that there are only two commodities: A consumption–investment good and labor. It will be convenient to describe the technology by a production function $F(k, l)$. To any amounts of capital investment $k \geq 0$ and of labor input $l \geq 0$, applied in the initial period, the production function assigns the *total* amount $F(k, l)$ of consumption–investment good available at the last period. Then

$$Y = \{(-k, -l, x, 0) \colon k \geq 0, l \geq 0, x \leq F(k, l)\} - \mathbb{R}^4_+.$$

Note that labor is a primary factor; that is, it cannot be produced. ■

Example 20.C.2: *Cost-of-Adjustment Model.* Suppose that there are three goods: capacity, a consumption good, and labor. With the amounts k and l of invested capacity and labor at the initial period, one gets $F(k, l)$ units of consumption good output at the last period. This output can be transformed into invested capacity at the last period at a cost of $k' + \gamma(k' - k)$ units of consumption good for k' units of capacity, where $\gamma(\cdot)$ is a convex function satisfying $\gamma(k' - k) = 0$ for $k' < k$ and $\gamma(k' - k) > 0$ for $k' > k$. The term $\gamma(k' - k)$ represents the cost of adjusting capacity upward in a given period relative to the previous period. (Note the marginal cost of doing so increases with invested capacity of the period.) Formally, the production set Y is

$$Y = \{(-k, 0, -l, k', x, 0) \colon k \geq 0, l \geq 0, k' \geq 0, x \leq F(k, l) - k' - \gamma(k' - k)\} - \mathbb{R}^6_+. \quad ■$$

5. See Ramsey (1928) and Solow (1956). The same model was also introduced in Swan (1956).

Example 20.C.3: *Two-Sector Model.* We could make a more general distinction between an investment and a consumption good than the one embodied in Examples 20.C.1 and 20.C.2. Indeed, we could let the production set be

$$Y = \{(-k, 0, -l, k', x, 0): k \geq 0, l \geq 0, k' \geq 0, x \leq G(k, l, k')\} - \mathbb{R}^6_+,$$

where k, k' are, respectively, the investments in the initial and the last periods. Note that the investment and the consumption good need not be perfectly substitutable [they are produced in two separate sectors, so to speak; see Uzawa (1964)]. If they are [i.e., if the transformation function $G(k, l, k')$ has the form $F(k, l) - k'$] then this example is equivalent to the Ramsey–Solow model of Example 20.C.1. If it has the form $G(k, l, k') = F(k, l) - k' - \gamma(k' - k)$ then we have the cost-of-adjustment model of Example 20.C.2. ■

Example 20.C.4: $(N + 1)$*-Sector Model.* As in Example 20.C.3, we have a consumption good and labor, but we now interpret k and k' as N-dimensional vectors. For simplicity of exposition, in Example 20.C.3 we have taken $G(k, l, k')$ to be defined for any $k \geq 0$, $k' \geq 0$. In general, however, this could lead to the production of negative amounts of consumption good. To avoid this it is convenient to complete the specification by means of an admissible domain A of (k, l, k') combinations. Then

$$Y = \{(-k, 0, -l, k', x, 0): (k, l, k') \in A \text{ and } x \leq G(k, l, k')\} - \mathbb{R}^{2(N+2)}_+. \qquad ■$$

Once we have specified our technology, we can define what constitutes a path of production plans.

Definition 20.C.1: The list $(y_0, y_1, \ldots, y_t, \ldots)$ is a *production path*, or *trajectory*, or *program*, if $y_t \in Y \subset \mathbb{R}^{2L}$ for every t.

Note that along a production path $(y_0, \ldots, y_t, \ldots)$ there is overlap in the time indices over which the production plans y_{t-1} and y_t are defined. Indeed, both $y_{a,t-1} \in \mathbb{R}^L$ and $y_{bt} \in \mathbb{R}^L$ represent plans, made respectively at dates $t - 1$ and t, for input use or output production at date t. Thus, we have, at every t, a net input–output vector equal to $y_{a,t-1} + y_{bt} \in \mathbb{R}^L$ (at $t = 0$, we put $y_{a,-1} = 0$; this convention is kept throughout the chapter).[6] The negative entries of this vector stand for amounts of inputs that have to be injected from the outside at period t if the path is to be realized, that is, amounts of input required at period t for the operation of y_{t-1} and y_t in excess of the amounts provided as outputs by the operation of y_{t-1} and y_t. Similarly, the positive entries represent the amounts of goods left over after input use and thus available for final consumption at time t.

The situation is entirely analogous to the description of the production side of an economy in Chapter 5. If we think of the technology at every t as being run by a distinct firm (or as an aggregate of distinct firms) and of $\hat{y}_t$ as an infinite sequence with nonzero entries (equal to y_t) only in the t and $t + 1$ places, then $\sum_t \hat{y}_t$ is the aggregate production path; and it is also precisely the sequence that assigns the net input–output vector $y_{a,t-1} + y_{bt} \in \mathbb{R}^L$ to period t. If we had a finite horizon, the current setting would thus be a particular case of the description of production in

6. A minor point of notation: when there is any possibility of confusion or ambiguity in the reading of indices, we insert commas; for example, we write $y_{a,t-1}$ instead of $y_{a\,t-1}$.

Chapter 5. With an infinite horizon there is a difference: we now have a countable infinity of commodities and of firms instead of only a finite number. As we shall see, this is not a minor difference. It will, however, be most helpful to arrange our discussion around the exploration of the analogy with the finite horizon case by asking the same questions we posed in Section 5.F regarding the relationship between efficient production plans and price equilibria.

Definition 20.C.2: The production path $(y_0, \ldots, y_t, \ldots)$ is *efficient* if there is no other production path $(y'_0, \ldots, y'_t, \ldots)$ such that

$$y_{a,t-1} + y_{bt} \le y'_{a,t-1} + y'_{bt} \quad \text{for all } t,$$

and equality does not hold for at least one t.

In words: the path $(y_0, \ldots, y_t, \ldots)$ is efficient if there is no way that we can produce at least as much final consumption in every period using at most the same amount of inputs in every period (with at least one inequality strict). The definition is exactly parallel to Definition 5.F.1.

What constitutes a *price vector* in the current intertemporal context? It is natural to define it as a sequence $(p_0, p_1, \ldots, p_t, \ldots)$, where $p_t \in \mathbb{R}^L$. For the moment we shall not ask where this sequence comes from. We assume that it is somehow given and that it is available to any possible production unit. The prices should be thought of as present-value prices. We shall discuss further the nature of these prices in the next section.

Given a path $(y_0, \ldots, y_t, \ldots)$ and a price sequence $(p_0, \ldots, p_t, \ldots)$, the profit level associated with the production plan at t is

$$p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}.$$

We now pursue the implications of profit maximization on the production plans made period by period.

Definition 20.C.3: The production path $(y_0, \ldots, y_t, \ldots)$ is *myopically*, or *short-run, profit maximizing for the price sequence* $(p_0, \ldots, p_t, \ldots)$ if for every t we have

$$p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \ge p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at} \quad \text{for all } y'_t \in Y.$$

Prices $(p_0, \ldots, p_t, \ldots)$ capable of sustaining a path $(y_1, \ldots, y_t, \ldots)$ as myopically profit-maximizing are often called *Malinvaud prices* for the path [because of Malinvaud (1953)].[7]

Does the first welfare theorem hold for myopic profit maximization? That is, if $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing with respect to strictly positive prices, does it follow that $(y_0, \ldots, y_t, \ldots)$ is efficient? In a finite-horizon economy this conclusion holds true because of Proposition 5.F.1, but a little thought reveals that in the infinite-horizon context it need not. The intuition for a negative answer rests on the phenomenon of *capital overaccumulation*. Suppose that prices increase through

7. Observe that we do not require that $\sum_t p_t \cdot (y_{a,t-1} + y_{bt}) < \infty$. In principle, a production path may have an infinite present value. We saw in Sections 5.E and 5.F, where we had a finite number of commodities and firms that individual, decentralized profit maximization and overall profit maximization amounted to the same thing. Because of the possibility of an infinite present value, the existence of a countable number of commodities *and* production sets makes this a more delicate matter in the current context. See Exercises 20.C.2 to 20.C.5 for a discussion.

time fast enough. Then it may very well happen that at every single period it always pays to invest everything at hand. Along such a path, consumption never takes place—hardly an efficient outcome.

Example 20.C.5: With $L = 1$, let $Y = \{(-k, k'): k \geq 0, k' \leq k\} \subset \mathbb{R}^2$. This is just a trivial storage technology. Consider the path where $y_t = (-1, 1)$ for all t; that is, we always carry forward one unit of good. Then $y_{a,-1} + y_{b0} = -1$ and $y_{a,t-1} + y_{bt} = 0$ for all $t > 0$. This is not efficient; just consider the path $y'_t = (0, 0)$ for all t, which has $y'_{a,t-1} + y'_{bt} = 0$ for all $t \geq 0$. But for the stationary price sequence where $p_t = 1$ for all t, $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing. ■

Efficiency will obtain if, in addition to myopic profit maximization, the (present) value of the production path becomes insignificant as $t \to \infty$. Precisely, efficiency obtains if the (present) value of the period t production plan for period $t + 1$ goes to zero, that is, if $p_{t+1} \cdot y_{at} \to 0$ as $t \to \infty$. This is the so-called *transversality condition*. Note that the condition is violated in the storage illustration of Example 20.C.5.

Proposition 20.C.1: Suppose that the production path $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing with respect to the price sequence $(p_0, \ldots, p_t, \ldots) \gg 0$. Suppose also that the production path and the price sequence satisfy the *transversality condition* $p_{t+1} \cdot y_{at} \to 0$. Then the path $(y_0, \ldots, y_t, \ldots)$ is efficient.

Proof: Suppose that the path $(y'_0, \ldots, y'_t, \ldots)$ is such that $y_{a,t-1} + y_{bt} \leq y'_{a,t-1} + y'_{bt}$ for all t, with equality not holding for at least one t. Then there is $\varepsilon > 0$ such that if we take a T sufficiently large for some strict inequality to correspond to a date previous to T, we must have

$$\sum_{t=0}^{T} p_t \cdot (y'_{a,t-1} + y'_{bt}) > \sum_{t=0}^{T} p_t \cdot (y_{a,t-1} + y_{bt}) + \varepsilon.$$

In fact, if T is very large then $p_{T+1} \cdot y_{aT}$ is very small (because of the transversality condition) and therefore

$$\sum_{t=0}^{T} p_t \cdot (y'_{a,t-1} + y'_{bt}) > p_{T+1} \cdot y_{aT} + \sum_{t=0}^{T} p_t \cdot (y_{a,t-1} + y_{bt}).$$

By rearranging terms—a standard trick in dynamic economics—this can be rewritten as (recall the convention $y_{a,-1} = y'_{a,-1} = 0$)

$$p_T \cdot y'_{bT} + \sum_{t=0}^{T-1} (p_{t+1} \cdot y'_{at} + p_t \cdot y'_{bt}) > \sum_{t=0}^{T} (p_{t+1} \cdot y_{at} + p_t \cdot y_{bt}).$$

We must thus have either $p_{t+1} \cdot y'_{at} + p_t \cdot y'_{bt} > p_{t+1} \cdot y_{at} + p_t \cdot y_{bt}$ for some $t \leq T - 1$ or $p_T \cdot y'_{bT} > p_{T+1} \cdot y_{aT} + p_T \cdot y_{bT}$. In either case we obtain a violation of the myopic profit-maximization assumption [recall that by the possibility of truncation we have $(y'_{bT}, 0) \in Y$]. Therefore, no such path $(y'_0, \ldots, y'_t, \ldots)$ can exist.

Note that the essence of the argument is very simple. The key fact is that if the transversality condition holds, then for T large enough we can approximate the overall profits of the truncated path $(y_0, \ldots, y_T)$ by the sum of the net values of period-by-period input output realizations (up to period T). It does not matter whether we match the inputs and the outputs per period or per firm (that is, "per production plan"). If the horizon is far enough away, either method will come down to Profits = Total Revenue − Total Cost. ■

Proposition 20.C.1 tells us that a modified version of the first welfare theorem holds in the dynamic production setting. Let us now ask about the second welfare theorem: *Given an efficient path* $(y_0, \ldots, y_t, \ldots)$, *can it be price supported?* In Proposition 5.F.2 we gave a positive answer to this question which applies to the finite-horizon case. In the current infinite-horizon situation we could decompose the question into two parts:

(i) *Is there a system of Malinvaud prices* $(p_0, \ldots, p_t, \ldots)$ *for* $(y_0, \ldots, y_t, \ldots)$, that is, a sequence $(p_0, \ldots, p_t, \ldots)$ with respect to which $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing?

(ii) *If the answer to* (i) *is yes, can we conclude that the pair* $(y_0, \ldots, y_t, \ldots)$, $(p_0, \ldots, p_t, \ldots)$ *satisfies the transversality condition?*

The answer to (ii) *is "not necessarily."* In Section 20.E we will see, by means of an example, that the transversality condition is definitely not a necessary property of Malinvaud prices.

The answer to (i) *is "Essentially yes."* We illustrate the matter by means of two examples and then conclude this section by a small-type discussion of the general situation.

Example 20.C.6: *Ramsey–Solow Model Continued.* In this model, we can summarize a path by the sequence (k_t, l_t, c_t) of total capital usage, labor usage, and amount available for consumption. From now on we assume that $k_{t+1} + c_{t+1} = F(k_t, l_t)$ and that the sequence l_t of labor inputs is exogenously given. Then it is enough to specify the capital path $(k_0, \ldots, k_t, \ldots)$. Denoting by (q_t, w_t) the prices of the two commodities at t, we have that profits at t are $q_{t+1}F(k_t, l_t) - q_t k_t - w_t l_t$ and, therefore, the necessary and sufficient conditions for short-run profit maximization at t are

$$\frac{q_t}{q_{t+1}} = \nabla_1 F(k_t, l_t) \qquad \text{and} \qquad \frac{w_t}{q_{t+1}} = \nabla_2 F(k_t, l_t).$$

Note that, up to a normalization (we could put $q_0 = 1$), these first-order conditions determine supporting prices for *any* feasible capital path (see Exercise 20.C.6).

The transversality condition says that $q_{t+1}F(k_t, l_t) \to 0$. If the sequence of productions $F(k_t, l_t)$ is bounded, then it suffices that $q_t \to 0$. In view of Proposition 20.C.1, we can conclude that a set of sufficient conditions for efficiency of a feasible and bounded capital path $(k_0, \ldots, k_t, \ldots)$ is that there exist a sequence of output prices $(q_0, \ldots, q_t, \ldots)$ such that

$$\frac{q_t}{q_{t+1}} = \nabla_1 F(k_t, l_t) \qquad \text{for all } t \tag{20.C.1}$$

and

$$q_t \to 0 \qquad (\text{equivalently, } 1/q_t \to \infty). \tag{20.C.2}$$

Because of the possibility of capital overaccumulation, (20.C.1), which is necessary, is not alone sufficient for efficiency. On the other hand, (20.C.2) is not necessary (see Section 20.E). Cass (1972) obtained a weakened version of (20.C.2) that, with (20.C.1), is both necessary and sufficient.[8] The condition is

$$\sum_{t=0}^{\infty} \frac{1}{q_t} = \infty. \tag{20.C.2'}$$

■

Example 20.C.7: *Cost of Adjustment Model continued.* In the cost of adjustment model, a production plan at time $t-1$ involves the variables k_{t-1}, l_{t-1}, k_t, c_t. We associate with these variables the prices q_{t-1}, w_{t-1}, q_t, s_t. Profits are then

$$s_t(F(k_{t-1}, l_{t-1}) - k_t - \gamma(k_t - k_{t-1})) + q_t k_t - q_{t-1}k_{t-1} - w_{t-1}l_{t-1}.$$

Using the first-order profit-maximization conditions with respect to k_t and k_{t-1} we get the following two conditions:

(i) $q_t = s_t(1 + \gamma'(k_t - k_{t-1}))$; that is, the price of capacity at t equals the investment cost in extra capacity at t.

(ii) $q_{t-1} = s_t(\nabla_1 F(k_{t-1}, l_{t-1}) + \gamma'(k_t - k_{t-1}))$; that is, the price of capacity at $t-1$ equals the return at t of one unit of extra capacity at $t-1$ (the return has two parts: the increased production at t and the saving in the cost of capacity adjustment at t).

Combining (i) and (ii),

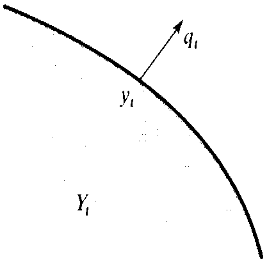
$$\frac{q_{t-1}}{q_t} = \frac{\nabla_1 F(k_{t-1}, l_{t-1}) + \gamma'(k_t - k_{t-1})}{1 + \gamma'(k_t - k_{t-1})}. \tag{20.C.3}$$

Note that if there are no adjustment costs [i.e., if $\gamma(\cdot)$ is identically equal to zero], then (20.C.3) is precisely (20.C.1). Observe also that, in parallel to Example 20.C.6, condition (20.C.3) determines short-run supporting prices for any feasible capital path. ■

In a general smooth model it is not difficult to explain how the supporting prices $(p_0, \ldots, p_t, \ldots)$ for an efficient path $(y_0, \ldots, y_t, \ldots)$ can be constructed. Note that, because of efficiency, every y_t belongs to the boundary of Y. The smoothness property that we require is that, for every t, the production set Y has a single (normalized) outward normal $q_t = (q_{bt}, q_{at})$ at y_t (we could, for example, normalize q_t to have unit length); see Figure 20.C.1. Less geometrically, smoothness means that at $y_t \in Y$ all the marginal rates of transformation (MRT) of inputs for inputs, inputs for outputs, and outputs for outputs are uniquely defined.

We claim that the efficiency property implies that for every t we have that $q_{a,t-1} = \beta q_{bt}$ for some $\beta > 0$. Heuristically: for any two commodities their MRT as outputs at t for the production decision taken at time $t-1$ must be the same as their MRT as inputs at t for the production decision taken at time t. If this were not so, it would be possible to generate a surplus of goods. The argument is standard (recall the analysis of Section 16.F). Consider, for example, Figure 20.C.2, where in panel (a) we have drawn the output transformation frontier through $y_{a,t-1}$ (i.e., keeping $y_{b,t-1}$ fixed) and in panel (b) the input isoquant through y_{bt} (i.e., keeping y_{at} fixed; recall the sign conventions for inputs). We see that if the slopes at these points are not the same, then it is possible to move from $y_{a,t-1}$ to $y'_{a,t-1}$ and from y_{bt} to y'_{bt} in such a way that $y'_{a,t-1} + y'_{bt} > y_{a,t-1} + y_{bt}$, thus contradicting efficiency.

8. Some additional, very minor, regularity conditions on the production function $F(\cdot)$ are required for the validity of this equivalence.



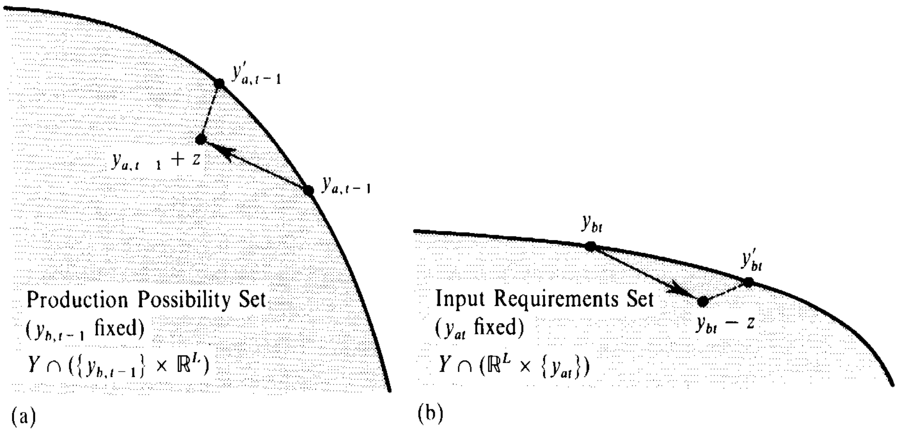


Figure 20.C.1 (left) Smooth production set.

Figure 20.C.2 (right) A production path that is inefficient at T.

We construct the desired price sequence $(p_0, \ldots, p_t, \ldots)$ by induction. Put $p_0 = q_{b0}$ (i.e., the relative prices at $t = 0$ are the MRTs between goods at the initial part of the production plan $y_0 \in Y \subset \mathbb{R}^{2L}$). Suppose now that the prices $(p_0, \ldots, p_T)$ have already been determined, and that every y_t up to $t = T - 1$ is myopically profit maximizing for these prices. Because of the first-order conditions for profit maximization at $T - 1$, we have that $p_T = \alpha q_{a,T-1}$ for some $\alpha > 0$. We know that $q_{a,T-1} = \beta q_{bT}$ for some $\beta > 0$. Then $p_T = \alpha\beta q_{bT}$. Therefore, if we put $p_{T+1} = \alpha\beta q_{aT}$, we have that $(p_T, p_{T+1}) = (\alpha\beta q_{bT}, \alpha\beta q_{aT})$ is proportional to $q_T = (q_{bT}, q_{aT})$, which means that y_T is profit maximizing for (p_T, p_{T+1}). Hence we have extended our sequence to $(p_0, \ldots, p_{T+1})$ and we can keep going.

Note that, as in Examples 20.C.6 and 20.C.7, the construction of the supporting short-run prices does not make full use of the efficiency. What is used is that the production path is "short-run efficient" (that is, the production path cannot be shown inefficient by changes in the production plans at a finite number of dates).

The above observations can be made into a perfectly rigorous argument for the existence of Malinvaud prices in the smooth case. The proof for the nonsmooth case is more complex. It must combine an appeal to the separating hyperplane theorem (to get prices for truncated horizons) with a limit operation as the horizon goes to infinity. With a minor technical condition (call *nontightness* in the literature), this limit operation can be carried out.

20.D Equilibrium: The One-Consumer Case

In this section, we bring the consumption and the production sides together and begin the study of equilibrium in the intertemporal setting. We shall start with the one-consumer case. As we will see in Section 20.G, the relevance of this case goes beyond the domain of applicability of the representative consumer theory of Chapter 4.

An economy is specified by a *short-term production technology* $Y \subset \mathbb{R}^{2L}$, *a utility function* $u(\cdot)$ defined on $\mathbb{R}^L_+$, a *discount factor* $\delta < 1$, and, finally, a (bounded) sequence of *initial endowments* $(\omega_0, \ldots, \omega_t, \ldots)$, $\omega_t \in \mathbb{R}^L_+$.

We assume that Y *satisfies hypotheses* (i) to (iv) of Section 20.C and that $u(\cdot)$ *is strictly concave, differentiable, and has strictly positive marginal utilities throughout its domain.*

Prices are given to us as sequences $(p_0, \ldots, p_t, \ldots)$ with $p_t \in \mathbb{R}^L_+$. As in Chapter 19 we can interpret these prices either as the prices of a complete system of forward markets occurring simultaneously at $t = 0$ or as the correctly anticipated (present value) prices of a sequence of spot markets. We will consider only bounded price sequences. In fact, most of the time we will have $\|p_t\| \to 0$.[9]

Given a production path $(y_0, \ldots, y_t, \ldots)$, $y_t \in Y$, the induced stream of consumptions $(c_0, \ldots, c_t, \ldots)$ is given by

$$c_t = y_{a,t-1} + y_{bt} + \omega_t.$$

If $c_t \geq 0$ for every t, then we say that the production path $(y_0, \ldots, y_t, \ldots)$ is *feasible*: Given the initial endowment stream the production path is capable of sustaining nonnegative consumptions at every period.

To keep the exposition manageable *from now on we restrict all our production paths and consumption streams to be bounded.* Delicate points come up in the general case, which are better avoided in a first approach. Alternatively, we could simply assume that our technology is such that any feasible production path is bounded.

Given a production path $(y_0, \ldots, y_t, \ldots)$ and a price sequence $(p_0, \ldots, p_t, \ldots)$, the induced stream of profits $(\pi_0, \ldots, \pi_t, \ldots)$ is given by

$$\pi_t = p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \qquad \text{for every } t.$$

Fixing T and rearranging the terms of $\sum_{t \leq T} p_t \cdot c_t = \sum_{t \leq T} p_t \cdot (y_{a,t-1} + y_{bt} + \omega_t)$ we get

$$\sum_{t \leq T} (\pi_t + p_t \cdot \omega_t) - \sum_{t \leq T} p_t \cdot c_t = p_{T+1} \cdot y_{aT} \qquad \text{(20.D.1)}$$

Expression (20.D.1) is an important identity. It tells us that *the transversality condition is equivalent to the overall value of consumption not being strictly inferior to wealth* (i.e., there is no escape of purchasing power at infinity).

The definition of a Walrasian equilibrium is now as in the previous chapters. One only has to make sure that a few infinite sums make sense.

Definition 20.D.1: The (bounded) production path $(y_0^*, \ldots, y_t^*, \ldots)$, $y_t^* \in Y$, and the (bounded) price sequence $p = (p_0, \ldots, p_t, \ldots)$ constitute a *Walrasian* (or *competitive*) equilibrium if:

(i) $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t \geq 0$ for all t. (20.D.2)

(ii) For every t,

$$\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^* \geq p_t \cdot y_b + p_{t+1} \cdot y_a \qquad \text{(20.D.3)}$$

for all $y = (y_b, y_a) \in Y$.

(iii) The consumption sequence $(c_0^*, \ldots, c_t^*, \ldots) \geq 0$ solves the problem

$$\begin{aligned} \text{Max } & \sum_t \delta^t u(c_t) \\ \text{s.t. } & \textstyle\sum_t p_t \cdot c_t \leq \sum_t \pi_t + \sum_t p_t \cdot \omega_t. \end{aligned} \qquad \text{(20.D.4)}$$

Condition (i) is the *feasibility* requirement. Condition (ii) is the short-run, or myopic, profit-maximization condition already considered in Section 20.C (Definition 20.C.3). The form of the budget constraint in part (iii) deserves comment. Note first that there is a single budget constraint. As in Chapter 19, this amounts to an assumption of *completeness*, which means, in one interpretation, that at time $t = 0$

9. Keep in mind that prices are to be thought of as measured in current-value terms.

there is a forward market for every commodity at every date, or, in another, that assets (e.g., money) are available that are capable of transferring purchasing power through time (see Exercise 20.D.1 for more on this). Secondly, observe that the strict monotonicity of $u(\cdot)$ implies that if we have reached utility maximization then, a fortiori, total wealth (denoted w) must be finite; that is,

$$w = \sum_t \pi_t + \sum_t p_t \cdot \omega_t < \infty.$$

Moreover, at the equilibrium consumptions the budget constraint of (20.D.4) must hold with equality.

An important consequence of the last observation is that at equilibrium the transversality condition is satisfied. Formally, we have Proposition 20.D.1.

Proposition 20.D.1: Suppose that the (bounded) production path $(y_0^*, \ldots, y_t^*, \ldots)$ and the (bounded) price sequence $(p_0, \ldots, p_t, \ldots)$ constitute a Walrasian equilibrium. Then the transversality condition $p_{t+1} \cdot y_{at}^* \to 0$ holds.

Proof: Denote $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t$. By expression (20.D.1) we have

$$\sum_{t \leq T} (\pi_t + p_t \cdot \omega_t) - \sum_{t \leq T} p_t \cdot c_t = p_{T+1} \cdot y_{aT}.$$

Since each of the sums in the left-hand side converges to $w < \infty$ as $T \to \infty$, we conclude that $p_{T+1} \cdot y_{aT}^* \to 0$. ■

Another implication of $w < \infty$ is the possibility of replacing condition (ii) of Definition 20.D.1 by

(ii′) The production path $(y_0^*, \ldots, y_t^*, \ldots)$ maximizes total profits, in the sense that for any other path $(y_0, \ldots, y_t, \ldots)$ and any T we have

$$\sum_{t=0}^{t=T} (p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}) \leq \sum_t (p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^*) < \infty.$$

Clearly, (ii′) implies (ii), and (ii) with $w < \infty$ implies (ii′) (see Exercise 20.D.2). Thus, at equilibrium prices, the test of myopic and of overall profit maximization coincide. Could a similar statement be made for an appropriate concept of myopic utility maximization? We now investigate this question.

Definition 20.D.2: We say that the consumption stream $(c_0, \ldots, c_t, \ldots)$ is *myopically, or short-run, utility maximizing* in the budget set determined by $(p_0, \ldots, p_t, \ldots)$ and $w < \infty$ if utility cannot be increased by a new consumption stream that merely transfers purchasing power between some two consecutive periods.

The key fact is presented in Exercise 20.D.3.

Exercise 20.D.3: Show that a consumption stream $(c_0, \ldots, c_t, \ldots) \gg 0$ is short-run utility maximizing for $p = (p_0, \ldots, p_t, \ldots)$ and $w < \infty$ if and only if it satisfies $\sum_t p_t \cdot c_t = w$ and the collection of first-order conditions:

For every t there is $\lambda_t > 0$ such that

$$\lambda_t p_t = \nabla u(c_t) \quad \text{and} \quad \lambda_t p_{t+1} = \delta \nabla u(c_{t+1}). \tag{20.D.5}$$

It follows from (20.D.5) that $\lambda_t p_t = \nabla u(c_t)$ and $\lambda_{t-1} p_t = \delta \nabla u(c_t)$. Therefore, $\lambda_{t-1} = \delta \lambda_t$ and so $\lambda_0 = \delta^t \lambda_t$. Hence letting $\lambda = \lambda_0$, we see that (20.D.5) is actually equivalent to

$$\text{For some } \lambda, \qquad \lambda p_t = \delta^t \nabla u(c_t) \quad \text{for all } t. \tag{20.D.6}$$

Once we realize that myopic utility maximization in a budget set amounts to (20.D.6), we can verify that overall utility maximization follows. This is done in Proposition 20.D.2.

Proposition 20.D.2: If the consumption stream $(c_0, \dots, c_t, \dots)$ satisfies $\sum_t p_t \cdot c_t = w < \infty$ and condition (20.D.6), then it is utility maximizing in the budget set determined by $(p_0, \dots, p_t, \dots)$ and w.

Proof: We first note that we cannot improve upon $(c_0, \dots, c_t, \dots)$ by transferring purchasing power only through a finite number of dates. Indeed, (20.D.6) implies that the first-order sufficient conditions for any such constrained utility maximization problem are satisfied.

Suppose now that $(c'_0, \dots, c'_t, \dots)$ is a consumption stream satisfying the budget constraint and yielding higher total utility. Then for a sufficiently large T, consider the stream $(c''_0, \dots, c''_t, \dots)$ with $c''_t = c'_t$ for $t \le T$ and $c''_t = c_t$ for $t > T$. Because $\delta < 1$, there is $\varepsilon > 0$ such that if T is large enough then there is an improvement of utility of more than 2ε in going from $(c_0, \dots, c_t, \dots)$ to $(c''_0, \dots, c''_t, \dots)$. Since $w < \infty$, the amount $\sum_{t > T} |p_t \cdot (c_t - c'_t)|$ can be made arbitrarily small. Hence, for large T the stream $(c''_0, \dots, c''_t, \dots)$ is almost budget feasible. It follows that it can be made budget feasible by a small sacrifice of consumption in the first period resulting in a utility loss not larger than ε. Overall, it still results in an improvement. But this yields a contradiction because only the consumption in a finite number of periods has been altered in the process. ■

Example 20.D.1: In this example we illustrate the use of conditions (20.D.6) for the computation of equilibrium prices. Suppose that we are in a one-commodity world with utility function $\sum_t \delta^t \ln c_t$. Given a price sequence $(p_0, \dots, p_t, \dots)$ and wealth w, the first-order conditions for utility maximization (20.D.6) are

$$\lambda p_t = \frac{\delta^t}{c_t} \quad \text{for all } t, \quad \text{and} \quad \sum_t p_t c_t = w.$$

Hence, $w = \sum_t p_t c_t = (1/\lambda) \sum_t \delta^t = (1/\lambda)[1/(1 - \delta)]$ and so $p_t c_t = \delta^t/\lambda = \delta^t(1 - \delta)w$ for all t. Note that this implies a *constant rate of savings* because $p_T c_T / (\sum_{t \ge T} p_t c_t) = 1 - \delta$, for all T (Exercise 20.D.4).[10]

We now discuss three possible production scenarios.

(i) The economy is of the exchange type; that is, there is no possibility of production and we are given an initial endowment sequence $(\omega_0, \dots, \omega_t, \dots) \gg 0$. Then the equilibrium must involve $c_t^* = \omega_t$ for every t, and therefore, normalizing to $\sum_t p_t \omega_t = 1$, the equilibrium prices should be

$$p_t = \frac{\delta^t (1 - \delta)}{\omega_t} \qquad \text{for every } t.$$

10. Logarithmic utility functions facilitate computation and are very important in applications. However, they are not continuous at the boundary ($\ln c_t \to -\infty$ as $c_t \to 0$) and therefore violate one of our maintained assumptions. This does not affect the current analysis but should be kept in mind.

(ii) Suppose instead that $\omega_0 = 1$ and $\omega_t = 0$ for $t > 0$. There is, however, a linear production technology transforming every unit of input at t into $\alpha > 0$ units of output at $t + 1$. Because of the boundary behavior of the utility function, consumption will be positive in every period, and therefore the technology will be in operation at every period. The linearity of the technologies then has the important implication that the equilibrium price sequence is completely determined by the technology. Putting $p_0 = 1$, we must have $p_t = 1/\alpha^t$. Wealth is $w = p_0\omega_0 = 1$, and therefore the equilibrium consumptions must be $c_t^* = [\delta^t(1-\delta)]/p_t = (\alpha\delta)^t(1-\delta)$. Note that, as long as $1 \leq \alpha < 1/\delta$, both the price and the consumption sequences are bounded. Observe also the interesting fact that for this example we have been able to compute the equilibrium without explicitly solving for the sequence of capital investments.

(iii) We are as in (ii) except that we now have a general technology $F(k)$ transforming every unit k_t, of investment at t into $F(k_t)$ units of output at $t + 1$. This output can then be used indistinctly for consumption or investment purposes at $t + 1$. That is, $c_{t+1} = F(k_t) - k_{t+1}$. The logarithmic form of the utility function allows for a shortcut to the computation of equilibrium prices. Indeed, say that $(p_0, \ldots, p_t, \ldots)$ are equilibrium prices and $(c_0^*, \ldots, c_t^*, \ldots)$, $(k_0^*, \ldots, k_t^*, \ldots)$ equilibrium paths of consumption and capital investment. Then we know that at any T a constant fraction δ of remaining wealth is invested. That is,

$$p_{T+1}k_{T+1}^* = \delta\left(\sum_{t \geq T+1} p_t c_t^*\right) = \delta p_{T+1}F(k_t^*).$$

Therefore, we must have $k_{t+1}^* = \delta F(k_t^*)$ for every t. With $k_0 = \omega_0 = 1$ given, this allows us to iteratively compute the sequence of equilibrium capital investments. The sequence of prices is then obtained from the profit-maximization conditions $p_{t+1}F'(k_t^*) - p_t = 0$. ■

Since a Walrasian equilibrium is myopically profit maximizing and satisfies the transversality condition (Proposition 20.D.1), we know from Proposition 20.C.1 that it is production efficient (assuming $p_t \gg 0$ for all t). Can we strengthen this to the claim that the full first welfare theorem holds? We will now verify that we can. In the current one-consumer problem, Pareto optimality simply means that the equilibrium solves the utility-maximization problem under the technological and endowment constraints:

$$\text{Max} \sum_t \delta^t u(c_t), \tag{20.D.7}$$

$$\text{s.t. } c_t = y_{a,t-1} + y_{bt} + \omega_t \geq 0 \quad \text{and} \quad y_t \in Y \text{ for all } t.$$

Proposition 20.D.3: Any Walrasian equilibrium path $(y_0^*, \ldots, y_t^*, \ldots)$ solves the planning problem (20.D.7).

Proof: Denote by B the budget set determined by the Walrasian equilibrium price sequence $(p_0, \ldots, p_t, \ldots)$ and wealth $w = \sum_t \pi_t + \sum_t p_t \cdot \omega_t$, where

$$\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{a,t+1}^*$$

for all t. That is,

$$B = \{(c'_0, \ldots, c'_t, \ldots): c'_t \geq 0 \text{ for all } t \text{ and } \textstyle\sum_t p_t \cdot c'_t \leq w\}.$$

By the definition of Walrasian equilibrium, the utility of the stream $(c_0^*, \ldots, c_t^*, \ldots)$ defined by $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t$ is maximal in this budget set. It suffices, therefore, to show that any feasible path $(y''_0, \ldots, y''_t, \ldots)$, that is, any path for which $y''_t \in Y$ and $c''_t = y''_{a,t-1} + y''_{bt} + \omega_t \geq 0$ for all t, must yield a consumption stream in B. To see this note that, for any T,

$$\sum_{t \leq T} p_t \cdot c''_t = \sum_{t \leq T-1} (p_t \cdot y''_{bt} + p_{t+1} \cdot y''_{at}) + p_T \cdot y''_{bT} + \sum_{t \leq T} p_t \cdot \omega_t.$$

By the possibility of truncation of production plans, we have $(y''_{bT}, 0) \in Y$. Therefore, by short-run profit maximization, $p_t \cdot y''_{bT} \leq \pi_T$ and $p_t \cdot y''_{bt} + p_{t+1} \cdot y''_{at} \leq \pi_t$ for all $t \leq T - 1$. Hence,

$$\sum_{t \leq T} p_t \cdot c''_t \leq \sum_{t \leq T} \pi_t + \sum_{t \leq T} p_t \cdot \omega_t \leq w \quad \text{for all } T,$$

which implies $\sum_t p_t \cdot c''_t \leq w$. ■

Let us now ask for the converse of Proposition 20.D.3 (i.e., for the second welfare theorem question; see chapter 16): Is any solution $(y_0, \ldots, y_t, \ldots)$ to the planning problem (20.D.7) a Walrasian equilibrium? In essence, the answer is "yes," but the precise theorems are somewhat technical because, to obtain a well-behaved price system (i.e., a price system as we understand it: a sequence of nonzero prices), one needs some regularity condition on the path. We give an example of one such result.[11]

Proposition 20.D.4: Suppose that the (bounded) path $(y_0^*, \ldots, y_t^*, \ldots)$ solves the planning problem (20.D.7) and that it yields strictly positive consumption (in the sense that, for some $\varepsilon > 0$, $c_{\ell t} = y_{\ell a,t-1}^* + y_{\ell bt}^* + \omega_{\ell t} > \varepsilon$ for all ℓ and t). Then the path is a Walrasian equilibrium with respect to some price sequence $(p_0, \ldots, p_t, \ldots)$.

Proof: We provide only a sketch of the proof. A possible candidate for an equilibrium price system is suggested by expression (20.D.6):

$$p_t = \delta^t \nabla u(c_t^*) \qquad \text{for all } t,$$

where $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t$. Because $(c_0^*, \ldots, c_t^*, \ldots)$ is bounded above and bounded away from the boundary (uniformly in t) we have $\sum_t \|p_t\| < \infty$, which implies the transversality condition. In turn, by expression (20.D.1) this yields $\sum_t p_t \cdot c_t^* = \sum_t (\pi_t + p_t \cdot \omega_t) = w < \infty$. Therefore, by Proposition 20.D.2, the utility-maximization condition holds.

It remains to establish that short-run profit maximization also holds. To that effect suppose that this is not so, that is, that for some T there is $y' \in Y$ with

$$p_T \cdot y'_b + p_{T+1} \cdot y'_a > p_T \cdot y_{bT}^* + p_{T+1} \cdot y_{aT}^* = \pi_T.$$

Let $(y'_1, \ldots, y'_t, \ldots)$ be the path with $y'_T = y'$ and $y'_t = y_t^*$ for any $t \neq T$. Let $(c'_0, \ldots, c'_t, \ldots)$ be the associated consumption stream. Because of the convexity of Y and the strict positivity property of $(c_0^*, \ldots, c_t^*, \ldots)$ we can assume that $y'_T = y'$ is sufficiently close to y_T^* for us to

11. A general treatment would involve, as in Sections 15.C or 16.D, the application of a suitable version (here infinite-dimensional) of the separating hyperplane theorem. The next result gets around this by exploiting the differentiability of $u(\cdot)$. It is thus parallel to the discussion in Section 16.F.

have $c'_t \gg 0$ for all t and, moreover, for it to be legitimate to determine the sign of

$$\sum_t \delta^t(u(c'_t) - u(c^*_t)) = \delta^T(u(c'_T) - u(c^*_T)) + \delta^{T+1}(u(c'_{T+1}) - u(c^*_{T+1}))$$

by signing the first-order term

$$\begin{aligned}\delta^T \nabla u(c^*_T)\cdot(c'_T - c^*_T) &+ \delta^{T+1} \nabla u(c^*_{T+1})\cdot(c'_{T+1} - c^*_{T+1}) \\ &= p_T\cdot(y'_{bT} - y^*_{bT}) + p_{T+1}\cdot(y'_{aT} - y^*_{aT}) \\ &= p_T\cdot y'_{bT} + p_{T+1}\cdot y'_{aT} - p_T\cdot y^*_{bT} - p_{T+1}\cdot y^*_{aT} > 0.\end{aligned}$$

But this conclusion contradicts the assumption that $(y^*_0, \ldots, y^*_t, \ldots)$ solves (20.D.7). ■

The close connection between the solutions of the equilibrium and the planning problem (20.D.7) has three important implications for, respectively, the existence, uniqueness, and computation of equilibria.

The first implication is that it reduces the question of the *existence* of an equilibrium to the possibility of solving a single optimization problem, albeit an infinite-dimensional one.

Proposition 20.D.5: Suppose that there is a uniform bound on the consumption streams generated by all the feasible paths. Then the planning problem (20.D.7) attains a maximum; that is, there is a feasible path that yields utility at least as large as the utility corresponding to any other feasible paths.

The proof, which is purely technical and which we skip, involves simply establishing that, in a suitable infinite-dimensional sense, the objective function of problem (20.D.7) is continuous and the constraint set is compact.

The second implication is that it allows us to assert the *uniqueness* of equilibrium.

Proposition 20.D.6: The planning problem (20.D.7) has at most one consumption stream solution.

Proof: The proof consists of the usual argument showing that the maximum of a strictly concave function in a convex set is unique. Suppose that $(y_0, \ldots, y_t, \ldots)$ and $(y'_0, \ldots, y'_t, \ldots)$ are feasible paths with $\sum_t \delta^t u(c_t) = \sum_t \delta^t u(c'_t) = \gamma$, where $(c_0, \ldots, c_t, \ldots)$ and $(c'_0, \ldots, c'_t, \ldots)$ are the consumption streams associated with the two production paths. Consider $y''_t = \frac{1}{2}y_t + \frac{1}{2}y'_t$. Then the path $(y''_0, \ldots, y''_t, \ldots)$ is feasible and at every t the consumption level is $c''_t = \frac{1}{2}c_t + \frac{1}{2}c'_t$. Hence, $\sum_t \delta^t u(c''_t) \geq \gamma$, with the inequality strict if $c_t \neq c'_t$ for some t. Thus, if $c_t \neq c'_t$ for some t, the paths $(y_0, \ldots, y_t, \ldots)$, $(y'_0, \ldots, y'_t, \ldots)$ could not both solve (20.D.7). ■

The third implication is that Proposition 20.D.3 provides a workable approach to the *computation* of the equilibrium. We devote the rest of this section to elaborating on this point.

The Computation of Equilibrium and Euler Equations

It will be convenient to pursue the discussion of computational issues in the slightly restricted setting of Example 20.C.4, the $(N + 1)$-sector model. To recall, we have N capital goods, labor, and a consumption good. We fix the endowments of labor to a constant level through time. A function $G(k, k')$ gives the total amount of consumption good obtainable at any t if the investment in capital goods at $t - 1$ is given by the vector $k \in \mathbb{R}^N$, the investment at t is required to be $k' \in \mathbb{R}^N_+$, and the labor usage at $t-1$ and t is fixed at the level exogenously given by the initial endowments. We denote by $A \subset \mathbb{R}^N \times \mathbb{R}^N$ the region of pairs $(k, k') \in \mathbb{R}^{2N}$ compatible with nonnegative consumption [i.e., $A = \{(k, k') \in \mathbb{R}^{2N}: G(k, k') \geq 0\}$]. For notational convenience, we write $u(G(k, k'))$ as $u(k, k')$. We assume that A is convex and that $u(\cdot, \cdot)$ is strictly concave. Also, at $t = 0$ there is some already installed capital investment $\bar{k}_0$ and this is the only initial endowment of capital in the economy.

In this economy the planning problem (20.D.7) becomes[12]

$$\text{Max} \quad \sum_t \delta^t u(k_{t-1}, k_t) \qquad (20.D.8)$$

$$\text{s.t. } (k_{t-1}, k_t) \in A \text{ for every } t, \text{ and } k_0 = \bar{k}_0.$$

From now on we assume that (20.D.8) has a (bounded) solution. Because of the strict concavity of $u(\cdot, \cdot)$ this solution is unique.

For every $t \geq 1$ the vector of variables $k_t \in \mathbb{R}^N$ enters the objective function of (20.D.8) only through the two-term sum $\delta^t u(k_{t-1}, k_t) + \delta^{t+1} u(k_t, k_{t+1})$. Therefore, differentiating with respect to these N variables, we obtain the following necessary conditions for an interior path $(k_0, \ldots, k_t, \ldots)$ to be a solution of the problem (20.D.8):[13]

$$\frac{\partial u(k_{t-1}, k_t)}{\partial k'_n} + \delta \frac{\partial u(k_t, k_{t+1})}{\partial k_n} = 0 \qquad \text{for every } n \leq N \text{ and } t \geq 1.$$

In vector notation,

$$\nabla_2 u(k_{t-1}, k_t) + \delta\, \nabla_1 u(k_t, k_{t+1}) = 0 \qquad \text{for every } t \geq 1. \qquad (20.D.9)$$

Conditions (20.D.9) are known as the *Euler equations* of the problem (20.D.8).

Example 20.D.2: Consider the Ramsey–Solow technology of Example 20.C.1 (with $l_t = 1$ for all t). Then, $u(k, k') = u(F(k) - k')$ and $A = \{(k, k'): k' \leq F(k)\}$. Therefore, the Euler equations take the form

$$-u'(F(k_{t-1}) - k_t) + \delta u'(F(k_t) - k_{t+1})F'(k_t) = 0, \quad \text{for all } t \geq 1$$

or

$$\frac{u'(c_t)}{\delta u'(c_{t+1})} = F'(k_t) \qquad \text{for all } t \geq 1.$$

In words: the marginal utilities of consuming at t or of investing and postponing consumption one period are the same. ■

Example 20.D.3: Consider the cost-of-adjustment technology of Example 20.C.2 (except that as in Example 20.D.2 we fix $l_t = 1$ for all t and drop labor as an explicitly considered commodity) and suppose we have an overall firm that tries to maximize the infinite discounted sum of profits by means of a suitable investment policy in capacity. Output can be sold at a constant unitary price that, with a constant rate

12. By convention we put $u(k_{-1}, k_0) = 0$.

13. The expression "interior path" means that (k_t, k_{t+1}) is in the interior of A for all t. For the interpretation of the expression to come, recall also that k_n and k'_n stand, respectively, for the nth and the $(N + n)$th argument of $u(k, k')$.

of interest, gives a present value price of δ^t. Thus the problem becomes that of maximizing $\sum_t \delta^t[F(k_{t-1}) - k_t - \gamma(k_t - k_{t-1})]$. The Euler equations are then

$$-1-\gamma'(k_t - k_{t-1}) + \delta[F'(k_t) + \gamma'(k_{t+1} - k_t)] = 0 \qquad \text{for all } t \geq 1.$$

In words: the marginal cost of a unit of investment in capacity at t equals the discounted value of the marginal product of capacity at t *plus* the marginal saving in the cost of capacity expansion at $t + 1$. Note that, iterating from $t = 1$, we get

$$1 + \gamma'(k_1 - k_0) = \sum_{t \geq 1} \delta^t(F'(k_t) - 1).$$

In words: At the optimum, the cost of investing in an extra unit of capacity at $t = 1$ equals the discounted sum of the marginal products of a *maintained* increase of a unit of capacity.[14] See Exercise 20.D.5 for more detail.[15] ■

Suppose that a path $(k_0, \ldots, k_t, \ldots)$ satisfies the Euler necessary equations (20.D.9). From their own definition, and the concavity of $u(\cdot, \cdot)$, it follows that the Euler equations are also sufficient to guarantee that the trajectory cannot be improved upon by a trajectory involving changes in a single k_t. In fact, the same is true if the changes are limited to any finite number of periods (see Exercise 20.D.6). Thus, we can say that the Euler equations are necessary and sufficient for short-run optimization. The question is then: Do the Euler equations (or, equivalently, short-run optimization) imply long-run optimization? We shall see that, under a regularity property on the path (related, in a manner we shall not make explicit, to the transversality condition[16]), they do.

We say that the path $(k_0, \ldots, k_t, \ldots)$ is *strictly interior* if it stays strictly away from the boundary of the admissible region A. [More precisely, the path is strictly interior if there is $\varepsilon > 0$ such that for every t there is an ε neighborhood of (k_t, k_{t+1}) entirely contained in A.]

Proposition 20.D.7: Suppose that the path $(\bar{k}_0, \ldots, k_t, \ldots)$ is bounded, is strictly interior, and satisfies the Euler equations (20.D.9). Then it solves the optimization problem (20.D.8).

Proof: The basic argument is familiar. If $(\bar{k}_0, \ldots, k_t, \ldots)$ does not solve (20.D.8), then there is a feasible trajectory $(\bar{k}_0, \ldots, k'_t, \ldots)$ that gives a higher utility. To simplify the reasoning suppose that this trajectory is bounded. Then, by the concavity of the objective function, the boundedness of $(\bar{k}_0, \ldots, k_t, \ldots)$ and its strict interiority, we can assume that, for every t, k'_t is so close to k_t that $(k'_t, k_{t+1}) \in A$. We can now take T large enough for $\sum_{t<T} \delta^t u(k'_{t-1}, k'_t) > \sum_{t<T} \delta^t u(k_{t-1}, k_t)$ and define then a new trajectory $(\bar{k}_0, \ldots, k''_t, \ldots)$ by

14. That is to say, the extra unit of capacity available at $t = 1$ produces $F'(k_1)$ at $t = 2$. Of this amount, one unit is devoted to additional investment at $t = 2$. With this, at $t = 2$ the net addition of capacity has not changed (the initial and final capacities at $t = 2$ expand by one unit) and therefore there is no change in the adjustment cost paid. Consequently, the net gain at $t = 2$ in terms of commodity is $F'(k_1) - 1$. But this is not all the gain because the extra unit of capacity available at $t = 2$ produces $F'(k_2)$ at $t = 3$, and so on.

15. The ideas of this example are related to what is known in macroeconomic theory as the q-theory of investment. See, for example, Chapter 2 of Blanchard and Fischer (1989).

16. We refer to the storage illustration of Example 20.C.5 for the need to appeal to a regularity property.

$k_t'' = k_t'$ for $t \leq T$ and $k_t'' = k_t$ for $t > T$. The new trajectory is admissible [note that $(k_T', k_{T+1}) \in A$]; it coincides with $(\bar{k}_0, \ldots, k_t', \ldots)$ up to T and with $(\bar{k}_0, \ldots, k_t, \ldots)$ after T. Moreover, if T is large enough, it still gives higher utility than $(\bar{k}_0, \ldots, k_t, \ldots)$. But this is impossible because, as we have already indicated, the Euler equations imply short-run optimization, that is, they are the first-order conditions for the optimization problem where we are constrained to adjust only the variables corresponding to a finite number of periods (see Exercise 20.D.6). ■

It may be helpful at this stage to introduce the concept of the *value function* $V(k)$ and the *policy function* $\psi(k)$. Given an initial condition $k_0 = k$, the maximum value attained by (20.D.8) is denoted $V(k)$, and if $(k_0, k_1, \ldots, k_t, \ldots)$ is the (unique) trajectory solving (20.D.8) with $k_0 = k$, then we put $\psi(k) = k_1$. That is, $\psi(k) \in \mathbb{R}^N$ is the vector of optimal levels of investment, hence of capital, at $t = 1$ when the levels of capital at $t = 0$ are given by k.

What accounts for the importance of the policy function is the observation that if the path $(\hat{k}_0, \ldots, \hat{k}_t, \ldots)$ solves (20.D.8) for $k_0 = \hat{k}_0$ then, for any T, the path $(\hat{k}_T, \ldots, \hat{k}_{T+t}, \ldots)$ solves (20.D.8) for $k_0 = \hat{k}_T$. Thus, if $(k_0, \ldots, k_t, \ldots)$ solves (20.D.8) we must have

$$k_{t+1} = \psi(k_t) \text{ for every } t, \tag{20.D.10}$$

and we see that the optimal path can be computed from knowledge of k_0 and the policy function $\psi(\cdot)$. But how do we determine $\psi(\cdot)$? We now describe two approaches to the computation of $\psi(\cdot)$. The first exploits the Euler equations; the second rests on the method of *dynamic programming*.

The Euler equations (20.D.9) suggest an iterative procedure for the computation of $\psi(k)$. Fix $k_0 = k$ and consider the equations corresponding to k_1. With k_0 given, we have N equations in the $2N$ unknowns $k_1 \in \mathbb{R}^N$ and $k_2 \in \mathbb{R}^N$. There are therefore N degrees of freedom. Suppose that we try to fix k_1 arbitrarily [equivalently, we try to fix $-\nabla_2 u(k_0, k_1)$, the marginal costs of investment at $t = 1$] and then use the N Euler equations at $t = 1$ to solve for the remaining k_2 unknowns [equivalently, we adjust the commitments for investment at $t = 2$ so that the discounted marginal payoff of investment at $t = 1$, $\delta \nabla_1 u(k_1, k_2)$, equals the preestablished marginal cost of investment at $t = 1$, i.e. $-\nabla_2 u(k_0, k_1)$]. Suppose that such a solution k_2 is found [by the strict concavity of $u(\cdot)$, if there is one solution then it has to be unique]. We can then repeat the process. The N Euler equations for period 2 are now exactly determined: Both k_1 and k_2 are given, but we still have the N variables k_3 corresponding to $t = 3$ with which we can try to satisfy the N equations of period 2. Suppose that we reiterate in this fashion. There are three possibilities. The first is that the process breaks down somewhere, that is, that given k_{t-1} and k_t there is no solution k_{t+1} [or, more precisely, no solution with $(k_t, k_{t+1}) \in A$]; the second is that we generate a sequence that is unbounded (or nonstrictly interior); the third is that we generate a bounded (and strictly interior) sequence $(k_0, k_1, \ldots, k_t, \ldots)$. In the third case, by Proposition 20.D.7 we have obtained an optimum, and since by Proposition 20.D.6 the optimum is unique, we can conclude that *given k_0, the third possibility (the trajectory starting at k_0 and k_1 is strictly interior and bounded) can occur for at most one value of k_1. If it occurs, this value of k_1 is precisely $\psi(k_0)$*. Thus, the computational method is: Solve the difference equation induced by the Euler equations with initial condition (k_0, k_1) and then for fixed k_0 search for an initial condition k_1 generating a bounded infinite path.

Example 20.D.4: Consider a Ramsey–Solow model with linear technology $F(k) = 2k$ and utility function $\sum_t (1/2)^t \ln c_t$. Then $u(k_{t-1}, k_t) = \ln(2k_{t-1} - k_t)$ and the period-t Euler equation is (see Exercise 20.D.7)

$$k_{t+1} = 3k_t - 2k_{t-1}.$$

This difference equation has the solution $k_t = k_0 + (k_1 - k_0)(2^t - 1)$. If $k_1 < k_0$, then k_t eventually becomes negative. If $k_1 > k_0$, then k_t is unbounded. The only value of k_1 generating a bounded k_t is $k_1 = k_0$. Therefore, $\psi(k_0) = k_0$ for any k_0. It is instructive to see what happens if we try $k_1 \geq k_0$. Then, the path induced by the difference equation is feasible and, in fact, we have a constant level of consumption $c_t = 2k_{t-1} - k_t = 2k_0 - k_1$. Thus, for $k_1 > k_0$, we have here an example of a path that is compatible with the Euler equations but that is not optimal, because at $k_1 = k_0$ we get a higher level of constant consumption.[17] ■

The dynamic programming approach exploits the recursivity of the optimum problem (20.D.8), namely, the fact that

$$V(k) = \underset{k' \text{ with } (k,k') \in A}{\text{Max}} \quad u(k, k') + \delta V(k'), \tag{20.D.11}$$

and obtains $\psi(k)$ as the vector k' that solves (20.D.11). This, of course, only transforms the problem into one of computing the value function $V(\cdot)$. However, it turns out that, first, under some general conditions [e.g., if $V(\cdot)$ is bounded] the value function is the *only* function that solves (20.D.11) when viewed as a functional equation, that is, $V(\cdot)$ is the only function for which (20.D.11) is true for every k, and, second, that there are some well-known and quite effective algorithms for solving equations such as (20.D.11) for the unknown function $V(\cdot)$. (See Section M.M. of the Mathematical Appendix.)

We end this section by pointing out two implications of the definition of the value function (see Exercise 20.D.8):

(i) *The value function $V(k)$ is concave.*

(ii) *For every perturbation parameter $z \in \mathbb{R}^N$ with $(k + z, \psi(k)) \in A$ we have*

$$V(k + z) \geq u(k + z, \psi(k)) + \delta V(\psi(k)). \tag{20.D.12}$$

Suppose that $N = 1$ and $(k, \psi(k))$ is interior to A. For later reference we point out that from (i), (ii), and $V(k) = u(k, \psi(k)) + \delta V(\psi(k))$ we obtain

$$V'(k) = \nabla_1 u(k, \psi(k))$$

and, if $V(\cdot)$ is twice-differentiable,

$$V''(k) \geq \nabla^2_{11} u(k, \psi(k)).$$

(See Figure 20.D.1 and Exercise 20.D.9.[18])

17. Hence, when $k_1 > k_0$, the Euler equations lead to capital overaccumulation. We note, without further elaboration, that given a path satisfying the Euler equations we could use the equations themselves to determine a myopically supporting price sequence. However, if $k_1 > k_0$ this sequence will violate the transversality condition.

18. The expression $\nabla^2_{ij} f(\cdot)$ denotes the ij second partial derivative of the real-value function $f(\cdot)$.

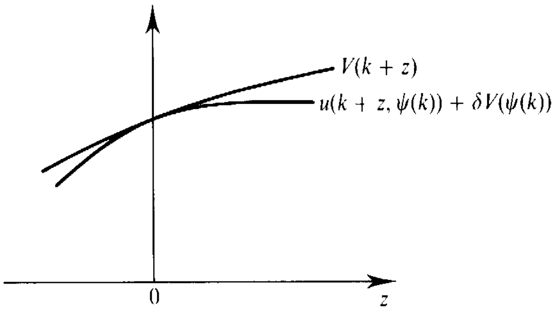


Figure 20.D.1
Along an optimal path the value function is majorized by the utilities of single-period adjustments.

20.E Stationary Paths, Interest Rates, and Golden Rules

In this section, we concentrate on the study of steady states. This study constitutes a first step towards the analysis of the dynamics of equilibrium paths. We refer to Bliss (1975), Gale (1973), or Weizsäcker (1971) for further analysis of steady-state theory.

We begin with a production set $Y \subset \mathbb{R}^{2L}$ satisfying the properties considered in Section 20.C. Recall that a production path is a sequence $(y_0, \ldots, y_t, \ldots)$ with $y_t \in Y$ for every t.

Definition 20.E.1: A production path $(y_0, \ldots, y_t, \ldots)$ is *stationary*, or a *steady state*, if there is a production plan $\bar{y} = (\bar{y}_b, \bar{y}_a) \in Y$ such that $y_t = \bar{y}$ for all $t > 0$.

Abusing terminology slightly, we refer to the "stationary path $(\bar{y}, \ldots, \bar{y}, \ldots)$" as simply the "stationary path $\bar{y}$."

The first important observation is that stationary paths *that are also efficient* are supportable by proportional prices.[19] This is shown in Proposition 20.E.1.

Proposition 20.E.1: Suppose that $\bar{y} \in Y$ defines a stationary and efficient path. Then, there is a price vector $p_0 \in \mathbb{R}^L$ and an $\alpha > 0$ such that the path is myopically profit maximizing for the price sequence $(p_0, \alpha p_0, \ldots, \alpha^t p_0, \ldots)$.

Proof: A complete proof is too delicate an affair, but the basic intuition may be grasped from the case in which production sets have smooth boundaries. For this case we can, in fact, show that *every* (myopically) supporting price sequence must be proportional.

By the efficiency of the path $(\bar{y}, \ldots, \bar{y}, \ldots)$, the vector $\bar{y}$ must lie at the boundary of Y. Let $\bar{q} = (\bar{q}_0, \bar{q}_1)$ be the unique (up to normalization) vector perpendicular to Y at $\bar{y}$. Also, by the small type discussion at the end of Section 20.C, there exists a price sequence $(p_0, \ldots, p_t, \ldots)$ that myopically supports this efficient path. Because $\bar{y} \in Y$ is short-run profit maximizing at every t we must have $(p_t, p_{t+1}) = \lambda_t(\bar{q}_0, \bar{q}_1)$ for some $\lambda_t > 0$. Therefore, $p_t = \lambda_t \bar{q}_0$ and $p_{t+1} = \lambda_t \bar{q}_1$ for all t. In particular, $p_t = \lambda_{t-1}\bar{q}_1$ and $p_{t+1} = \lambda_{t+1}\bar{q}_0$. Combining, we obtain $p_{t+1} = (\lambda_t/\lambda_{t-1})p_t$ and

19. To prevent possible misunderstanding, we warn that establishing the inefficiency of a given stationary path will typically require the consideration of nonstationary paths.

$p_{t+1} = (\lambda_{t+1}/\lambda_t)p_t$. From this we get $\lambda_t/\lambda_{t-1} = \lambda_{t+1}/\lambda_t$ for all $t \geq 1$. Hence, denoting this quotient by α, we have $p_{t+1} = \alpha p_t = \alpha^2 p_{t-1} = \cdots = \alpha^{t+1} p_0$.

The factor α has a simple interpretation. Indeed, $r = (1 - \alpha)/\alpha$ [so that $p_t = (1 + r)p_{t+1}$] can be viewed as a *rate of interest* implicit in the price sequence (see Exercise 20.E.1).

Proposition 20.E.1 is a sort of second welfare theorem result for stationary paths. We could also pose the parallel first welfare theorem question. Namely, suppose that $(\bar{y}, \ldots, \bar{y}, \ldots)$ is a stationary path myopically supported by a proportional price sequence with rate of interest r. If $r > 0$, then $p_t = (1/(1 + r))^t p_0 \to 0$ and therefore the transversality condition $p_t \cdot \bar{y}_a \to 0$ is satisfied. We conclude from Proposition 20.C.1 that the path is efficient. If $r \leq 0$, the transversality condition is not satisfied (p_t does not go to zero), but this does not automatically imply inefficiency because the transversality condition is sufficient but not necessary for efficiency. Suppose that $r < 0$ and, to make things simple, let us be in the smooth case again. Consider the stationary candidate paths defined by the constant production plan $\bar{y}_\varepsilon = (\bar{y}_b + \varepsilon e, \bar{y}_a - \varepsilon e)$, where $e = (1, \ldots, 1) \in \mathbb{R}^L$. This candidate path uses fewer inputs (or produces more outputs) at $t = 0$ and generates exactly the same net input output vector at every other t. Therefore, if for some $\varepsilon > 0$, the candidate path is in fact a feasible path; that is, if $\bar{y}_\varepsilon \in Y$, then the stationary path $\bar{y}$ is not efficient (it overaccumulates). But if Y has a smooth boundary at $\bar{y}$, the feasibility of $\bar{y}_\varepsilon$ for some $\varepsilon > 0$ can be tested by checking whether $\bar{y}_\varepsilon - \bar{y} = \varepsilon(e, -e)$ lies below the hyperplane determined by the supporting prices $(p_0, [1/(1 + r)]p_0)$. Evaluating, we have $\varepsilon(1 - 1/(1 + r))p_0 \cdot e < 0$, because $r < 0$. Conclusion: For ε small enough, the stationary path $\bar{y}$ is dominated by the stationary path $\bar{y}_\varepsilon$. We record these facts for later reference in Proposition 20.E.2.

Proposition 20.E.2: Suppose that the stationary path $(\bar{y}, \ldots, \bar{y} \ldots)$, $\bar{y} \in Y$, is myopically supported by proportional prices with rate of interest r, then the path is efficient if $r > 0$ and inefficient if $r < 0$.

We have not yet dealt with the case $r = 0$, which as we shall see, is very important.[20] We will later verify in a more specific setup that efficiency obtains in this case.

Let us now bring in the consumption side of the economy and consider *stationary equilibrium paths*. Assuming differentiability and interiority, a stationary path $(\bar{y}, \ldots, \bar{y}, \ldots)$ that is also an equilibrium can be supported only (up to a normalization) by the price sequence $p_t = \delta^t \nabla u(\bar{c})$, where $\bar{c} = \bar{y}_b + \bar{y}_a$; recall Proposition 20.D.4 and expression (20.D.6). That is, *a stationary equilibrium is supported by a price sequence embodying a proportionality factor equal to the discount factor* δ, or, equivalently, with rate of interest $r = (1 - \delta)/\delta$.

Definition 20.E.2: A stationary production path that is myopically supported by proportional prices $p_t = \alpha^t p_0$ with $\alpha = \delta$ is called a *modified golden rule path*. A stationary production path myopically supported by constant prices $p_t = p_0$ is called a *golden rule path*.

20. Note that 0 is the rate of growth implicit in the path $(\bar{y}, \ldots, \bar{y}, \ldots)$. In a more general treatment we could allow for a constant returns technology and for the production path to be proportional (but not necessarily stationary). Then Proposition 20.E.2 remains valid with 0 replaced by the corresponding rate of growth.

Depending on the technology and on the discount factor δ, there may be a single or there may be several modified golden rule paths (see the small-type discussion at the end of this section). But in any case we have just seen that a *stationary equilibrium path is necessarily a modified golden rule path.* Thus, we have the important implication that the *candidates for stationary equilibrium paths* $(\bar{y}, \dots, \bar{y}, \dots)$ *are completely determined by the technology and the discount factor and are independent of the utility function* $u(\cdot)$.

To pursue the analysis it will be much more convenient to reduce the level of abstraction. Consider an extremely simple case, the Ramsey–Solow model technology of Example 20.C.1. We study trajectories with $l_t = 1$ for all t (imagine that there is available one unit of labor at every point in time). We can then identify a production path with the sequence of capital investments $(k_0, \dots, k_t, \dots)$.

Given $(k_0, \dots, k_t, \dots)$, denote $r_t = \nabla_1 F(k_t, 1) - 1$. Thus, r_t is the *net* (i.e., after replacing capital) *marginal productivity of capital.* Suppose that $k_t > 0$ and that the sequence of output prices $(q_0, \dots, q_t, \dots)$ and wages $(w_0, \dots, w_t, \dots)$ myopically price supports the given path. Then, by the first-order condition for profit maximization, we have $q_{t+1}(1 + r_t) - q_t = 0$. Hence r_t is the output rate of interest at time t implicit in the output price sequence $(q_0, \dots, q_t, \dots)$.

Let us now focus on the stationary paths of this example. Any $k \geq 0$ fixed through time constitutes a *steady state.* With any such steady state we can associate a constant surplus level $c(k) = F(k, 1) - k$ and a rate of interest $r(k) = \nabla_1 F(k, 1) - 1$, also constant through time.[21] Therefore, the supporting price–wage sequence is

$$(q_t, w_t) = \left(\frac{1}{1 + r(k)}\right)^t (q_0, w_0), \qquad \text{with } \frac{w_0}{q_0} = \frac{\nabla_2 F(k, 1)}{\nabla_1 F(k, 1)}.$$

Denote by $w(k)$ the real wage w_0/q_0 so determined. It is instructive to analyze how the steady-state levels of consumption $c(k)$, the rate of interest $r(k)$, and the real wage $w(k)$ depend on k.

Let $\bar{k}$ be the level of capital at which the steady-state consumption level is maximized [i.e., $\bar{k}$ solves $\text{Max}\, F(k, 1) - k$]. Note that $\bar{k}$ is characterized by $r(\bar{k}) = \nabla_1 F(\bar{k}, 1) - 1 = 0$. Thus $\bar{k}$ is precisely the *golden rule* steady state. The construction is illustrated in Figure 20.E.1, where we also represent the modified golden rule k_δ [characterized by $r(k_\delta) = \nabla_1 F(k_\delta, 1) - 1 = (1 - \delta)/\delta$]. Observe that if $k < \bar{k}$ then $r(k) > 0$. As we saw in Proposition 20.E.2, $r(k) > 0$ implies that the steady state k is efficient (thus, in particular, the modified golden rule is efficient: it gives less consumption than the golden rule but it also uses less capital). Similarly, if $k > \bar{k}$ then $r(k) < 0$ and we have inefficiency of the steady state k. What about $\bar{k}$?[22] We now argue that *the golden rule steady state* $\bar{k}$ *is efficient.* A graphic proof will be quickest. Suppose we try to dominate the constant path $\bar{k}$ by starting with $k_0 < \bar{k}$, so that consumption at $t = 0$ is raised. Since the surplus at $t = 1$ must be at least

21. Thus, $c(k)$ is the amount of good constantly available through time and usable as a flow for consumption purposes.

22. Recall that the associated price sequence is constant and that the transversality condition is therefore violated.

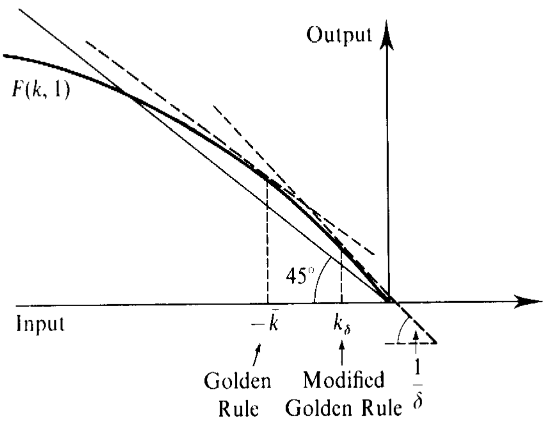


Figure 20.E.1
The production technology of the Ramsey–Solow model and the golden rule.

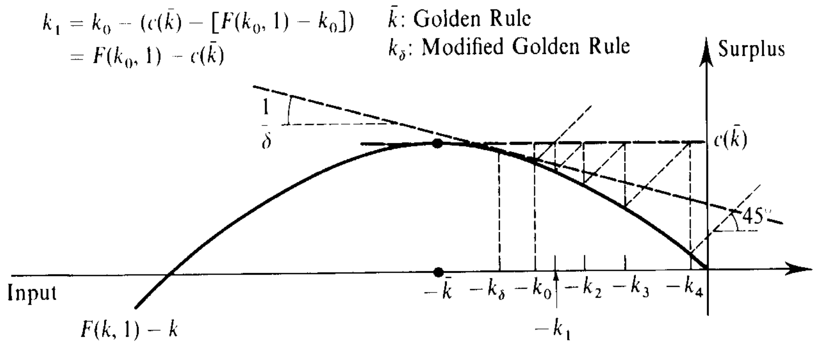


Figure 20.E.2
Ramsey–Solow model: the golden rule is efficient.

$c(\bar{k})$, the best we can do for k_1 is

$$k_1 = F(k_0, 1) - c(\bar{k}) = F(k_0, 1) - k_0 + k_0 - c(\bar{k}) < k_0,$$

because $F(k_0, 1) - k_0 < c(\bar{k})$. This new best possible value of k_1 is represented in Figure 20.E.2. In the figure we also see that as the process of determination of k_1 is iterated to obtain k_2, k_3 and so on we will, at some point get a $k_t < 0$. Hence, the path is not feasible, and we conclude that a constant $\bar{k}$ cannot be dominated from the point of view of efficiency: the attempt to use less capital at some stage will inexorably lead to capital depletion in finite time.

From the form of the production function, three "neoclassical" properties follow immediately (you are asked to prove them in Exercise 20.E.4):

(i) As k increases, the level $c(k)$ increases monotonically up to the golden rule level and then decreases monotonically.

(ii) The rate of interest $r(k)$ decreases monotonically with the level of capital k.

(iii) The real wage $w(k)$ increases monotonically with the level of capital. (For the validity of this property you should also assume that production function $F(k, l)$ is homogeneous of degree one.)

From the study of the steady states of the Ramsey–Solow model we have learnt at least six new things: First, the rate of interest is equal to the net marginal productivity of capital; second, the golden rule (i.e., zero rate of interest) path is characterized by a surplus-maximizing property among steady states; third, the golden rule is efficient; fourth, fifth, and sixth, we have the three neoclassical properties.

How general is all of this? That is, can we make similar claims for the general model with any number of goods? The answer, in short, is that the three neoclassical properties may or may not hold in a world with several capital goods, but the other three, duly interpreted, remain valid with great generality. Attempting to give proofs of all this would take us into too advanced material [see Bliss (1975) or Brock and Burmeister (1976)], but perhaps we can provide some intuition.

Suppose we consider the general $(N + 1)$-sector technology of Example 20.C.4. That is, $G(k, k')$ is the amount of consumption good available at any period if $k \in \mathbb{R}^N$ is the vector of levels of capital used in the previous period and the investment required in the period is $k' \in \mathbb{R}^N$ (we also let $l_t = 1$ for all t). At a steady-state path we have $k' = k$. Denote by $\hat{G}(k) = G(k, k)$ the level of consumption associated with the steady state k. If $G(\cdot, \cdot)$ is a concave function then so is $\hat{G}(\cdot)$. In particular, $\nabla\hat{G}(k) = 0$ characterizes the steady state with maximal level of consumption.

Consider a steady steady k. By Proposition 20.E.1, this steady state can be myopically supported by a proportional price sequence $s_t \in \mathbb{R}$, $q_t \in \mathbb{R}^N$. Here s_t is the price of the consumption good in period t, and q_t is the vector of prices of investment in period t. Because of proportionality there is an $r(k)$ such that $s_t = (1 + r(k))s_{t+1}$, $q_t = (1 + r(k))q_{t+1}$ for all t. Because of profit maximization,

$$\nabla_1 G(k, k) = \frac{1}{s_t} q_{t-1} \quad \text{and} \quad \nabla_2 G(k, k) = -\frac{1}{s_t} q_t \quad \text{for all } t \tag{20.E.1}$$

(you are asked to verify this in Exercise 20.E.5). Therefore,

$$\nabla\hat{G}(k) = \nabla_1 G(k, k) + \nabla_2 G(k, k) = \frac{1}{s_t}(q_{t-1} - q_t) = \frac{r(k)}{s_t} q_t,$$

that is, at any time *an extra dollar invested in a permanent increase of any capital good yields* $r(k)$ *dollars in extra value of (permanent) consumption.* In this precise sense the rate of interest measures the marginal productivity of capital. We see again that $\nabla\hat{G}(k) = 0$ (the necessary and sufficient condition for maximum steady-state consumption) is equivalent to $r(k) = 0$. Hence, the golden rule property holds: a steady-state level k yields maximal consumption across steady states if and only if it has associated with it a zero rate of interest. We add that we could also prove that the golden rule path is efficient.

As we have already indicated, the neoclassical properties do not carry over to the general setting. A taste of the possible difficulties can be given even if $N = 1$, that is, for the two-sector model of Example 20.C.3. In Figure 20.E.3 we represent the level curves of $G(k, k')$. The steady states correspond to the diagonal, where $k = k'$. Every steady state k can be myopically

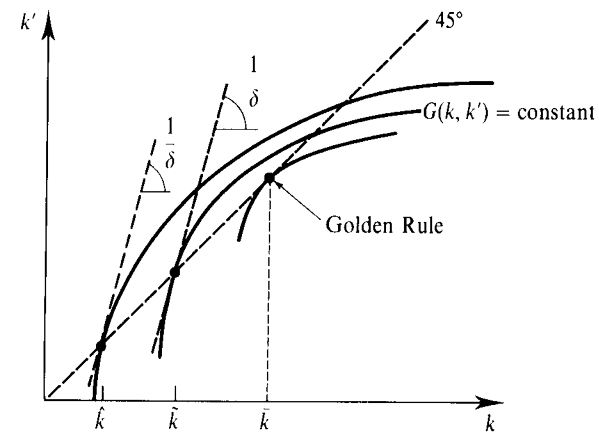


Figure 20.E.3
An example with several modified golden rules.

supported by proportional prices $q_t = (1 + r(k))q_{t+1}$ where, to insure profit maximization, q_t/q_{t+1} must be equal to the slope of the level curve through (k, k) (you should verify this in Exercise 20.E.6). Therefore, the efficient steady states, those with $r(k) \geq 0$, correspond to the subset of the diagonal that goes from the origin to the golden rule, where $r(\bar{k}) = 0$. In the special case of the Ramsey Solow model we have $G(k, k') = F(k, 1) - k'$ and therefore the level curves of $G(k, k')$ admit a quasilinear representation with respect to k' (i.e., they can be generated from each other by parallel displacement along the k' axis). In Exercise 20.E.7 you are asked to show that this guarantees the satisfaction of the neoclassical properties. In general, however, it is clear from Figure 20.E.3 that we may, for example, have two different $\hat{k}, \tilde{k} < \bar{k}$ such that, at the diagonal, the corresponding level curves have the same slope and therefore $r(\hat{k}) = r(\tilde{k})$ (contradicting the second neoclassical property). In particular, while the golden rule is unique [if the function $G(k, k')$ is strictly concave], there may be several modified golden rules [this is the case if, say, the discount factor δ is equal to the interest rate $r(\hat{k})$].

20.F Dynamics

In this section, we offer a few observations on the vast topic of the dynamic properties of equilibria. The basic framework is as in the previous section: a one-consumer economy with stationary technology and utility.

The arbitrarily given initial conditions[23] will typically not be compatible with a stationary equilibrium situation (e.g., the steady-state level of capital may be higher than the initial availability of capital). Therefore, the typical equilibrium path will be nonstationary. How complicated can the equilibrium dynamics be? Can we, for example, expect convergence to a modified golden rule? This would be nice, as it would tell us that our models carry definite long-run predictions.

We can gain much insight into these matters by considering a variation of the two-sector model of Example 20.C.3. We assume that the technology produces consumption goods (possibly of more than one kind) out of labor and a capital good. There is, as initial endowment, one unit of labor in each period, and we let $u(k, k')$ stand for the maximum utility that can be attained in any given period if in the previous period $k \in \mathbb{R}$ units of capital were installed and the current investment is required to be k' (and, in both periods, a unit of labor is used). There is a positive initial endowment of capital only at t = 0. Also, we take $u(\cdot, \cdot)$ to be strictly concave and differentiable.

We know from Proposition 20.D.3 and 20.D.4 that the equilibrium paths can be determined by means of the following planning problem:

$$\text{Max} \sum_t \delta^t \text{u}(k_{t-1}, k_t) \tag{20.F.1}$$

$$\text{s.t. } k_t \geq 0 \text{ and } k_0 = k \text{ is given.}$$

Suppose that $V(k)$ and $\psi(k)$ are value and policy functions, respectively, for the problem (20.F.1). These concepts were introduced in Section 20.D. As we explained there, the equilibrium dynamics are entirely determined by iterating the policy function [see expression (20.D.10)]. That is, given k_0, the equilibrium trajectory is

$$(k_0, k_1, k_2, \ldots) = (k_0, \psi(k_0), \psi(\psi(k_0)), \ldots).$$

23. That is, the initial endowment sequence $(\omega_0, \ldots, \omega_t, \ldots)$.

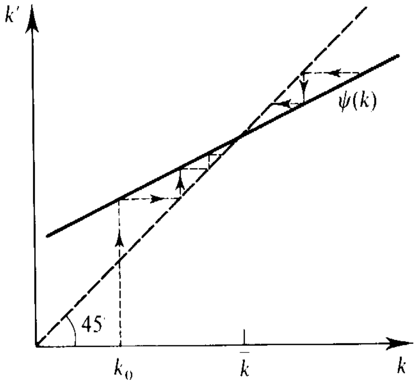


Figure 20.F.1 (left)
A single, stable steady state.

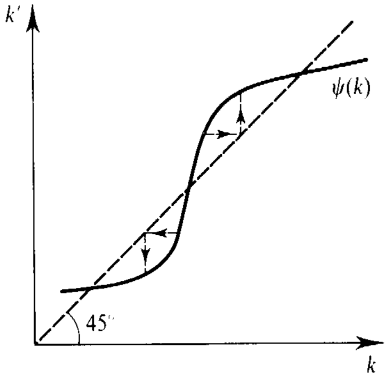


Figure 20.F.2 (right)
Several steady states, no cycles.

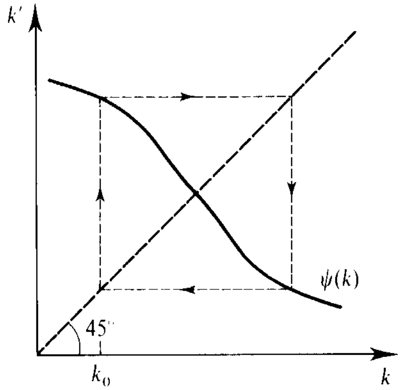


Figure 20.F.3 (left)
A single steady state and a cycle of period 2.

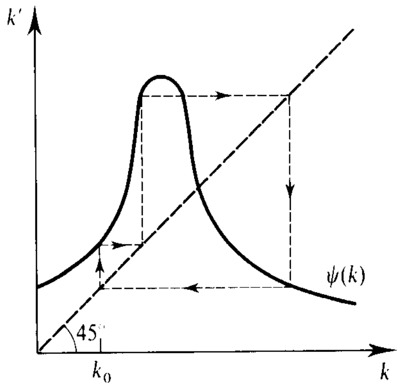


Figure 20.F.4 (right)
A cycle of period 3: chaos.

Note that a steady-state path $(\bar{k}, \ldots, \bar{k}, \ldots)$ is an equilibrium path (for $k_0 = \bar{k}$), and therefore a modified golden rule steady state path for discount factor δ (see Definition 20.E.2 and the discussion surrounding it), if and only if $\bar{k} = \psi(\bar{k})$.

Figures 20.F.1 through 20.F.4 represent four mathematical possibilities for this equilibrium dynamics. In Figure 20.F.1, we have the simplest possible situation: a monotonically increasing policy function with a single steady state $\bar{k}$. The steady state is then necessarily globally stable; that is, $k_t \to \bar{k}$ for any k_0. In Figure 20.F.2, the policy function is again monotonically increasing, but now there are several steady states. They have different stability properties, but it is still true that from any initial point we converge to some steady state. In Figure 20.F.3, the steady state is unique, but now the policy function is not increasing and cycles are possible. Finally, in Figure 20.F.4 we have a policy function that generates a cycle of period 3. It is known that a one-dimensional dynamical system exhibiting a nontrivial cycle of period 3 is necessarily *chaotic* [see Grandmont (1986) for an exposition of the mathematical theory]. We cannot go here into an explanation of the term "chaotic" in this context. It suffices to say that the equilibrium trajectory may wander in a complicated way and that its location in the distant future is very sensitive to initial conditions. The theoretical possibility of chaotic equilibrium trajectories is troublesome from the economic point of view. How is it to be expected that an auctioneer will succeed in computing them; or even worse, how would a consumer exactly anticipate such a sequence?

Unfortunately, the "anything goes" principle that haunted us in Chapter 17 in the form of the Sonnenschein–Mantel–Debreu theorem (Section 17.E) reemerges here in the guise of the Boldrin–Montruccio theorem [see Boldrin and Montruccio (1986)]: *Any candidate policy function $\psi(k)$ can be generated from some concave $u(k, k')$ and $\delta > 0$.* We will not state or demonstrate this theorem precisely, but the main idea of its proof is quite accessible. We devote the next few paragraphs to it.

Suppose for a moment that for a given $u(\cdot, \cdot)$ our candidate $\psi(\cdot)$ is such that $\psi(k)$ solves, for every k, the following "complete impatience" problem:

$$\underset{k' \geq 0}{\text{Max}} \quad u(k, k'). \tag{20.F.2}$$

This would be the problem of a decision maker who did not care about the future. While this is not quite the problem that we want to solve, it approximates it if we take $\delta > 0$ to be very low. Then the decision maker cares very little about the future and therefore its optimal action k' will, by continuity, be very close to $\psi(k)$. Hence, in an approximate sense, we are done if we can find a $u(\cdot, \cdot)$ such that $\psi(k)$ solves (20.F.2) for every k.

In order for a $\psi(k) > 0$ to solve (20.F.2), $u(k, \cdot)$ cannot be everywhere decreasing in its second argument (the optimal decision would then be $k' = 0$). In the simplest version of the Ramsey–Solow model (Example 20.C.1), the returns of k', the investment in the current period, accrue only in the next period, and therefore the utility function $u(k, k')$ is decreasing in k'. But in the current, more general, two-sector model there is no reason that forces this conclusion. Suppose, for example, that there are two consumption goods. The first is the usual consumption–investment good, while the second is a pure consumption good not perfectly substitutable with the first. Say that with an amount k of investment at time $t - 1$ one gets, jointly, k units of the consumption–investment good at time t and k units of the second consumption good at time $t - 1$. Accordingly, with k' units of the consumption–investment good invested at t one gets, jointly, k' units of the consumption–investment good at $t + 1$ and k' units of the second consumption good at t. Thus, if k and k' are the amounts of investment at $t - 1$ and t, respectively, then the bundle of consumption goods available at t is $(k - k', k')$. Hence, the utility function $u(\cdot, \cdot)$ has the form $u(k, k') = \hat{u}(k - k', k')$, where $\hat{u}(\cdot, \cdot)$ is a utility function for bundles of the two consumption goods.

Therefore, our problem is reduced to the following: Given $\psi(k)$ can we find $\hat{u}(\cdot, \cdot)$ such that $\psi(k)$ solves $\text{Max}_{k'}\ \hat{u}(k - k', k')$ for all k in some range? The problem is represented in Figure 20.F.5.[24] We see from the figure that the problem has formally become one of finding a concave utility function with a prespecified Engel curve at some given prices (in our case, the two prices are equal). Such a utility function can always be obtained. It is a well-known, and most plausible fact that the concavity of $\hat{u}(\cdot)$ imposes no restrictions on the shape that a single Engel curve may exhibit (see Exercise 20.F.1).

The news is not uniformly bad, however. In principle, as we have seen, everything may be possible; yet there are interesting and useful sufficient conditions implying a

24. We also assume that $\psi(k) < k$ for all k.

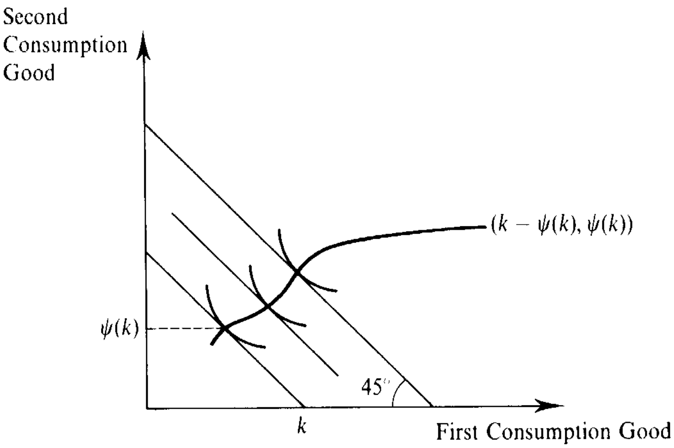


Figure 20.F.5 Construction of an arbitrary policy function in the completely impatient case.

well-behaved dynamic behavior. We discuss two types of conditions: a *low discount of time* and *cross derivatives of uniform positive sign.*

Low Discount of Time

One of the most general results of dynamic economics is the *turnpike theorem*, which, informally, asserts that *if the one-period utility function is strictly concave and the decision maker is very patient, then there is a single modified golden rule steady state that, moreover, attracts the optimal trajectories from any initial position.*

In the context of the two-sector model studied in this section, we can give some intuition for the turnpike theorem. Suppose that the value function $V(k)$, which is concave, is twice-differentiable.[25] At the end of Section 20.D, we saw that since by definition,

$$V(k + z) \geq u(k + z, \psi(k)) + \delta V(\psi(k))$$

for all z and k (with equality for $z = 0$), we must have

$$V'(k) = \nabla_1 u(k, \psi(k)) \quad \text{and} \quad V''(k) \geq \nabla^2_{11} u(k, \psi(k)) \quad \text{for all } k.$$

Also for all k, $\psi(k)$ solves the first-order condition

$$\nabla_2 u(k, \psi(k)) + \delta V'(\psi(k)) = 0. \tag{20.F.3}$$

Differentiating this first-order condition, we have (all the derivatives are evaluated at $k, \psi(k)$ and assumed to be nonzero)

$$\psi'(\cdot) = -\frac{\nabla^2_{21} u(\cdot)}{\nabla^2_{22} u(\cdot) + \delta V''(\cdot)}.$$

Because $\nabla^2_{22} u(\cdot) \leq 0$ and $\delta \nabla^2_{11} u(\cdot) \leq \delta V''(\cdot) \leq 0$, it follows that

$$|\psi'(\cdot)| \leq \left| \frac{\nabla^2_{21} u(\cdot)}{\nabla^2_{22} u(\cdot) + \delta \nabla^2_{11} u(\cdot)} \right|.$$

By the concavity of $u(\cdot)$ we have (see Sections M.C and M.D of the Mathematical Appendix)

$$(\nabla^2_{21} u(\cdot))^2 \leq \nabla^2_{11} u(\cdot)\, \nabla^2_{22} u(\cdot) < (\nabla^2_{11} u(\cdot) + \nabla^2_{22} u(\cdot))^2.$$

25. For a (very advanced) discussion of this assumption, see Santos (1991).

Hence, if the discount factor δ is close to 1, it is a plausible conclusion that $|\psi'(k)| < 1$ for all k. In technical language: $\psi(\cdot)$ is a *contraction*, and this implies global convergence to a unique steady state.[26] In Exercise 20.F.2 you are invited to draw the policy functions and the arrow diagrams for this case. A particular instance of a contraction is exhibited in Figure 20.F.1.

The turnpike theorem is valid for any number of goods. The precise statement and the proof of the theorem are subtle and technical [see McKenzie (1987) for a brief survey], but the main logic is simply conveyed. Consider the extreme case where there is complete patience, that is, "only the long-run matters." A difficulty is that it is not clear what this means for arbitrary paths; but at least for paths that are not too "wild," say for those that from some time become cyclical, it is natural to assume that it means that the paths are evaluated by taking the average utility over the cycle. Observe now that *for any cyclical nonconstant path, the strict concavity of the utility function implies that the constant path equal to the mean level of capital over the cycle yields a higher utility.* It may take some time to carry out a transition from the cycle to the constant path (e.g., it may be necessary to build up capital) but, as long as this can be done in a finite number of periods, the cost of the transition will not show up in the long run. Hence the cyclical nonconstant path cannot be optimal for a completely patient optimizer. By continuity, all this remains valid if δ is very close to 1. We can conclude, therefore, that if a path tends to a nonconstant cycle then we can always implement a finite transition to a suitable constant "long-run average," for a relatively large long-run gain of utility and a relatively low short-run cost. In fact, this conclusion remains valid whenever a path does not stabilize in the long-run. It follows that the optimal path must be asymptotically almost constant, which can only be the case if the path reaches and remains in a neighborhood of a modified golden rule steady state (recall from Section 20.E that those are the only constant paths that can be equilibria, and therefore optimal).[27]

Cross Derivative of Uniform Positive Sign

We shall concern ourselves here with the particular case of the two-sector model studied so far where $\nabla_1 u(k, k') > 0$ and $\nabla_2 u(k, k') < 0$ for all (k, k'). By a *cross derivative of uniform positive sign we mean that* $\nabla_{12} u(k, k') > 0$, again at *all points of the domain.* In words: An increase in investment requirements at one date leads to a situation of increased productivity (in terms of current utility) of the capital installed the previous date. Examples are the classical Ramsey–Solow model $u(F(k) - k')$ and the cost-of-adjustment model $u(F(k) - k' - \gamma(k' - k))$ (see Exercise 20.F.3). We shall argue that *under this cross derivative condition the policy function is increasing* (as in Figures 20.F.1 or 20.F.2), and therefore the optimal path converges to a stationary path.

To prove the claim, it is useful to express $\psi(k)$ as the k' solution to

$$\begin{aligned} \underset{(k', V)}{\text{Max}} \quad & u(k, k') + \delta V \\ \text{s.t. } & V \leq V(k'), \end{aligned} \tag{20.F.4}$$

26. We note that $\psi(\cdot)$ need not be monotone and the convergence may be cyclical, although the cycles will dampen through time.

27. Also, with δ close to 1, the modified golden rule will typically retain the uniqueness property of the golden rule.

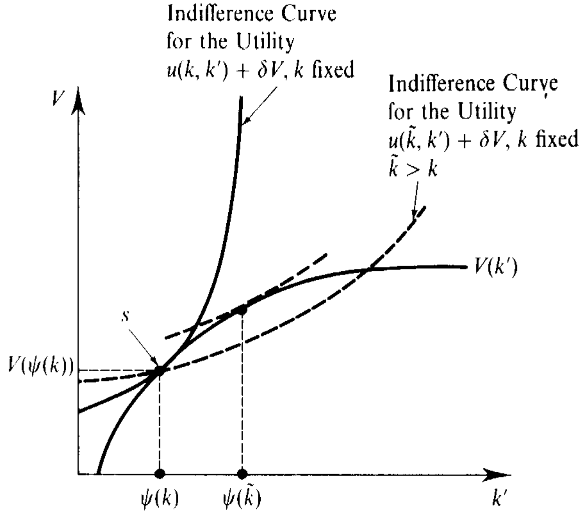


Figure 20.F.6
With the uniform positive sign cross derivative condition, the policy function is increasing.

where $V(\cdot)$ is the value function. For fixed k, problem (20.F.4) is represented in Figure 20.F.6. The marginal rate of substitution (MRS) between current investment k' and future utility V at $s = (\psi(k), V(\psi(k)))$ is $(1/\delta)\nabla_2 u(k, \psi(k)) < 0$. Suppose now that we take $\tilde{k} > k$. Then the indifference map in Figure 20.F.6 changes. Because $\nabla_{12} u(k, \psi(k)) > 0$, the MRS at s is altered in the manner displayed in the figure, that is, the indifference curve becomes flatter. But we can see then that necessarily $\psi(\tilde{k}) > \psi(k)$, as we wanted to show.

The cross derivative condition does not, by itself, imply the existence of a single modified golden rule. Thus, we could be in Figure 20.F.2 rather than in Figure 20.F.1. Note, however, that in many cases of interest it may be possible to show directly that the modified golden rule is unique. Thus, in both the classical Ramsey–Solow model of Example 20.C.1 and in the cost-of-adjustment model [with $\gamma'(0) = 0$] of Example 20.C.2, the modified golden rule is characterized by $F'(k) = 1/\delta$. Hence it is unique and, because the policy function is increasing, we conclude that every optimal path converges to it.

We also point out that if the cross derivative is of uniform *negative* sign, then, by the same arguments, $\psi(\cdot)$ is *decreasing*. While this allows for cycles, the dynamics are still relatively simple. In particular, the nonmonotonic shape associated with the possibility of chaotic paths (Figure 20.F.4) cannot rise. See Deneckere and Pelikan (1986) for more on these points.

Figure 20.F.6 is also helpful in illuminating the distinction between *transitory* and *permanent* shocks. One of the important uses of dynamic analysis in general, and of global convergence turnpike results in particular, is in the examination of how an economy at long-run rest reacts to a perturbation of the data at time $t = 1$. In an extremely crude classification, these perturbations can be of two types:

(i) *Transitory* shocks affect the environment of the economy only at $t = 1$; that is, they alter k_0 or, more generally, $u(k_0, \cdot)$, the utility function at $t = 1$. Then Figure 20.F.6 allows us to see how the equilibrium path will be displaced. The (k', V) indifference curve of $u(k_0, k') + \delta V$ changes, but the constraint function $V(k')$ remains unaltered. Hence, after the (transitory) shock, the new k_1^{tr} corresponds to the solution of the optimum problem depicted in Figure

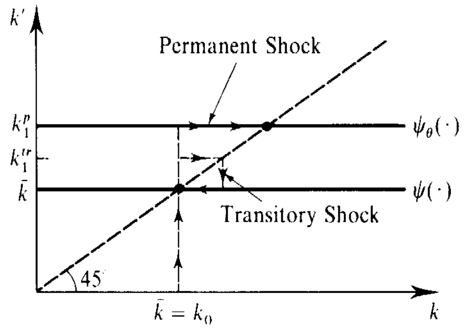


Figure 20.F.7
An example of dynamic adjustment under transitional and permanent shocks.

20.F.6 but with the new indifference map. From $t = 2$ on we simply follow the old policy function.

(ii) *Permanent* shocks move the economy to a new utility function $\hat{u}(k, k')$ constant over time. Then the entire policy function changes to a new $\hat{\psi}(\cdot)$. In terms of Figure 20.F.6 there would be a change in both the indifference curves *and* the constraint. The new k_1^p is now harder to determine and to compare with the preshock k_1 or, for the same shock at period 1, with k_1^{tr}; but it can often be done. We pursue the matter through Example 20.F.1.

Example 20.F.1: Consider the separable utility $u(k, k') = g(k) + h(k')$. This could be the investment problem of a firm: $g(k)$ is the maximal revenue obtainable with k, and $-h(k')$ is the cost of investment. Then $\nabla_{12}^2 u(k, k') = 0$ at all (k, k'). Our previous analysis of Figure 20.F.6, tells us that in this case $\psi(\cdot)$ is constant; that is, from any k_0 the economy goes in one step to its steady-state value $\bar{k}$. This is illustrated in Figure 20.F.7.

Suppose now there is a shock variable θ such that $u(k, k', \theta) = g(k, \theta) + h(k', \theta)$, with the preshock value being $\theta = 0$. The economy is initially at its steady state $\bar{k}$.

If there is a transitory shock to a small $\theta > 0$, then from the analysis of Figure 20.F.6 we can see that $k_1^{tr} \gtrless \bar{k}$ according to $\partial^2 h(\bar{k}, 0)/\partial k' \, \partial\theta \gtrless 0$. (Exercise 20.F.4 asks you to verify this.)

To evaluate the effects of a permanent shock to a small $\theta > 0$ (and therefore to a new $\psi_\theta(\cdot)$) the term

$$\partial^2 V(\bar{k}, 0)/\partial k \, \partial\theta = \partial^2 g(\bar{k}, 0)/\partial k \, \partial\theta$$

also matters [the previous equality follows from expression (20.F.3)]. Suppose, for example, that the shock is unambiguously favourable, in the sense that $\partial^2 g(\bar{k}, 0)/\partial k \, \partial\theta > 0$ and $\partial^2 h(\bar{k}, 0)/\partial k' \, \partial\theta > 0$. Then a careful analysis of Figure 20.F.6, would allow us to conclude that $k_1^p > k_1^{tr} > \bar{k}$. (Exercise 20.F.5 asks you to verify this. Note that the indifference map of Figure 20.F.6 is quasilinear with respect to V.) Figure 20.F.7 illustrates this case further. ■

20.G Equilibrium: Several Consumers

Up to now we have had a single consumer, or, to be more precise, a single type of consumer. The extension of the definition of equilibrium to economies with several consumers, say I, presents no particular difficulty. We simply have to rewrite Definition 20.D.1 as in Definition 20.G.1.

Definition 20.G.1: The (bounded) production path $(y_0^*, \ldots, y_t^*, \ldots)$, $y_t^* \in Y$, the (bounded) price sequence $(p_0, \ldots, p_t, \ldots) \geq 0$, and the consumption streams

$(c_{0i}^*, \ldots, c_{ti}^*, \ldots) \geq 0$, $i = 1, \ldots, I$, constitute a *Walrasian* (or *competitive*) equilibrium if:

(i)
$$\sum_i c_{ti}^* = y_{a,t-1}^* + y_{bt}^* + \sum_i \omega_{ti}, \text{ for all } t. \tag{20.G.1}$$

(ii) For every t,
$$\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^* \geq p_t \cdot y_{bt} + p_{t+1} \cdot y_{at} \tag{20.G.2}$$
for all $y = (y_{bt}, y_{at}) \in Y$.

(iii) For every i, the consumption stream $(c_{0i}^*, \ldots, c_{ti}^*, \ldots) \geq 0$ solves the problem
$$\begin{aligned} \text{Max} \quad & \sum_t \delta_i^t u_i(c_i) \\ \text{s.t.} \quad & \textstyle\sum_t p_t \cdot c_{ti} \leq \sum_t \theta_{ti} \pi_t + \sum_t p_t \cdot \omega_{ti} = w_i. \end{aligned} \tag{20.G.3}$$
where θ_{ti} is consumer i's given share of period t profits.

The first, and very important, observation to make is that, in complete analogy with the finite-horizon case (see Section 16.C), the first welfare theorem holds.[28]

Proposition 20.G.1: A Walrasian equilibrium allocation is Pareto optimal.

Proof: The proof is as in Proposition 16.C.1. Let the Walrasian equilibrium path under consideration be given by the production path $(y_0^*, \ldots, y_t^*, \ldots)$, the consumption streams $(c_{0i}^*, \ldots, c_{ti}^*, \ldots)$, $i = 1, \ldots, I$, and the price sequence $(p_0, \ldots, p_t, \ldots)$. Suppose now that the paths $(y_0, \ldots, y_t, \ldots)$, and $(c_{0i}, \ldots, c_{ti}, \ldots) \geq 0$, $i = 1, \ldots, I$, are feasible [i.e., they satisfy condition (i) of Definition 20.G.1] and are Pareto preferred to the Walrasian equilibrium.

By the utility-maximization condition we have $\sum_t p_t \cdot c_{ti} \geq w_i$ for all i, with at least one inequality strict. Hence,
$$\sum_t p_t \cdot \left(\sum_i c_{ti} \right) = \sum_i \left(\sum_t p_t \cdot c_{ti} \right) > \sum_i w_i. \tag{20.G.4}$$

Because of the profit maximization condition we get[29]
$$\begin{aligned} \sum_t p_t \cdot \left(\sum_i c_{ti} \right) &= \sum_t p_t \cdot \left(y_{a,t-1} + y_{bt} + \sum_i \omega_{ti} \right) \\ &= \sum_t p_t \cdot y_{a,t-1} + \sum_t p_t \cdot y_{bt} + \sum_t \sum_i p_t \cdot \omega_{ti} \\ &= \sum_{t \geq 1} (p_{t-1} \cdot y_{b,t-1} + p_t \cdot y_{a,t-1}) + \sum_i \sum_t p_t \cdot \omega_{ti} \\ &\leq \sum_t \pi_t + \sum_i \sum_t p_t \cdot \omega_{ti} = \sum_i w_i. \end{aligned}$$

But this conclusion contradicts (20.G.4). ■

28. Note also that, in the terminology of Chapter 19, the market structure is complete: Every consumer has a single budget constraint and, therefore, only prices limit the possibilities of transferring wealth across periods.

29. Recall that, by convention, $y_{a,-1} = 0$.

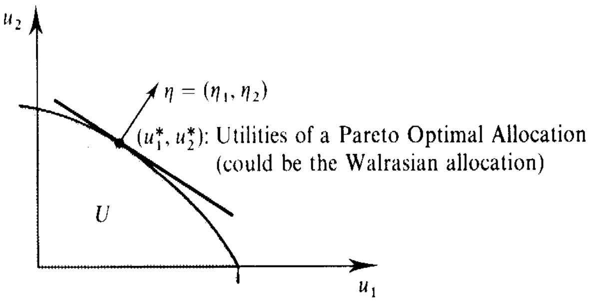


Figure 20.G.1
The Walrasian equilibrium as a solution of a planning problem.

We saw in Sections 16.E and 16.F that, under the assumption of concave utility functions, a Pareto optimal allocation of an economy with a finite number of commodities can be viewed as the solution of a planning problem. As described in Figure 20.G.1, the objective function of the planner is a weighted sum of the utilities of the different consumers (the weights being the reciprocal of the marginal utilities of wealth at the equilibrium with transfers associated with the particular Pareto optimum). The arguments of Section 16.E (in particular, Proposition 16.E.2) apply as well to the current infinite-horizon case. Therefore, Proposition 20.G.1 has, besides its substantive interest, a significant methodological implication. It tells us that the prices, productions, and aggregate consumptions of a given Walrasian equilibrium correspond exactly to those of a certain single-consumer economy. We give a more precise statement in Proposition 20.G.2. In it we restrict ourselves to the case of a common discount factor, namely, $\delta_i = \delta$ for all i.

Proposition 20.G.2: Suppose that $(y_0^*, \dots, y_t^*, \dots)$ is the production path and $(p_0, \dots, p_t, \dots)$ is the price sequence of a Walrasian equilibrium of an economy with I consumers. Then there are weights $(\eta_1, \dots, \eta_I) \gg 0$ such that $(y_0^*, \dots, y_t^*, \dots)$ and $(p_0, \dots, p_t, \dots)$ constitute a Walrasian equilibrium for the one-consumer economy defined by the utility $\sum_t \delta^t u(c_t)$, where $u(c_t)$ is the solution to $\text{Max} \sum_i \eta_i u_i(c_{ti})$ s.t. $\sum_i c_{ti} \le c_t$.

Proof: We will not give a rigorous proof, but the result is intuitive from Figure 20.G.1. From there we see (technically this involves, as in Proposition 16.E.2, an application of the separating hyperplane theorem) that there are weights $(\eta_1, \dots, \eta_I) \gg 0$ such that the equilibrium consumption streams maximize $\sum_i \eta_i (\sum_t \delta^t u_i(c_{ti}))$ over all feasible consumption streams, or, equivalently (it is here that the assumption of a common discount factor matters), the aggregate equilibrium consumption stream, solves the two-step planning problem specified by the definition of $u(c_t)$ and the maximization of $\sum_t \delta^t u(c_t)$. Because we already know (Proposition 20.D.4) that this is tantamount to the one-consumer equilibrium problem, we are done. ■

Proposition 20.G.2 allows us to conclude that the one-consumer theory developed in the last three sections remains highly relevant to the several-consumer case.[30] Somewhat informally, we can distinguish two types of properties of an equilibrium.

30. More generally, it remains highly relevant to any equilibrium model that guarantees the Pareto optimality of equilibria.

The *internal* properties are those that refer only to the structure of an equilibrium viewed solely in reference to itself (e.g., convergence to a steady state); the *external* properties refer to how the equilibrium relates to other possible equilibrium trajectories of the economy (e.g., uniqueness or local uniqueness). The message of Proposition 20.G.2 is that, because of Pareto optimality, the internal properties of an equilibrium of an economy with several consumers are those of its associated one-consumer economy. The implications of the one-consumer theory should not, however, be pushed beyond the internal properties. The reason is that *the weights defining the planning problem depend on the particular equilibrium considered.* For example, it is perfectly possible for there to be more than one equilibrium, each a Pareto optimum but supported by different weights.

What can be said about the determinacy properties of equilibrium; for example, about the finiteness of the number of equilibria? We will not be able to give a precise treatment of this matter, in part because it is very technical and in part because it is still an active area of research where the ultimate results may not yet be at hand. The basic intuition, however, can be transmitted. We begin by pointing out another implication of Proposition 20.G.1. Formally, our infinite-horizon model involves infinitely many variables (prices, say) and infinitely many equations (Euler equations, say). This is most unpleasant, as the mathematical theory described in Section 17.D applies only (and for good reasons, as we shall see in Section 20.H) to systems with a *finite* number of equations and unknowns. However, Proposition 20.G.1 allows us to view the equilibrium problem as one of finding not equilibrium prices but *equilibrium weights* η. If we do this then the equilibrium equations in our system are $I - 1$ in number, the same as the number of unknowns. More precisely, the ith equation would associate with the vector of weights $\eta = (\eta_1, \ldots, \eta_I)$, $\sum_i \eta_i = 1$, the wealth "gap" of consumer i:

$$\sum_t p_t(\eta)\cdot c_{ti}(\eta) - \sum_t (\theta_{ti}\pi_t(\eta) + p_t(\eta)\cdot\omega_{ti}) = 0,$$

where $p_t(\eta)$, $c_{ti}(\eta)$, and $\pi_t(\eta)$ correspond to the Pareto optimum indexed by η. See Appendix A of Chapter 17 for a construction similar to this. At any rate, once looked at as a wealth-equilibrating problem across a finite number of consumers, the central conjecture should be that, as in Chapter 17, the equilibrium set is nonempty and generically finite. That is, equilibrium exists and, except for pathological cases, there are only a finite number of weights solving the equilibrium equations (we could similarly go on to formulate an index theorem). Technical difficulties[31] aside, this central conjecture can be established in a wide variety of cases [see Exercise 20.G.3 and Kehoe and Levine (1985)].

We end this section with two remarks. The first derives from the question: Is there a relationship, a "correspondence," between internal and external properties? At least in a first approximation the answer is "no." We have seen that in a one-consumer economy the equilibrium is unique, but the equilibrium path may be complicated. Similarly, in a several-consumer economy there may be several equilibria, or even a continuum, each of them nicely converging to a steady state.[32]

31. These have to do with guaranteeing the differentiability of the relevant functions.

32. The simplest, trivial, example is the following. Suppose that $L = 2$, $I = 2$ and that there is no possibility of intertemporal production. Individual endowments are constant through time and the utility functions are concave. Then the intertemporal Walrasian equilibria correspond exactly to the infinite, constant repetitions of the one-period Walrasian equilibria (you are asked to prove this in Exercise 20.G.4). Because there are may be several of those, we obtain our conclusion.

The second remark brings home the point that Pareto optimality is key to an expectation of generic determinacy. Consider, as an example, a model of identical consumers but with an externality. The utility function, $u(k, k', e)$, now has three arguments: k and k' are the capital investments in the previous and the current periods, respectively, and e is the level of currently felt externality. Given, for the moment, an exogenously fixed externality path $(e_0, \ldots, e_t, \ldots)$, the (bounded, strictly interior) capital trajectory k_t is an equilibrium if it solves the planning problem for the utility functions $u(\cdot, \cdot, e_t)$, that is, if it satisfies the Euler equations:

$$\nabla_2 u(k_{t-1}, k_t, e_t) + \delta \nabla_1 u(k_t, k_{t+1}, e_{t+1}) = 0 \qquad \text{for all } t.$$

An overall equilibrium must take into account the technology determining the externality. Say that this is $e_t = k_t$; that is, the externality is a side product of current investment. Hence, the equilibrium conditions are

$$\nabla_2 u(k_{t-1}, k_t, k_t) + \delta \nabla_1 u(k_t, k_{t+1}, k_{t+1}) = 0 \qquad \text{for all } t. \tag{20.G.5}$$

Suppose that starting from an equilibrium steady state ($k_t = \bar{k}$ for all t), we try, as we did in Section 20.D, to generate a different equilibrium by fixing $k_0 = \bar{k}$, taking k_1 to be slightly different from $\bar{k}$, and then iteratively solving (20.G.5) for k_{t+1}. A sufficient (but not necessary) condition for this method to succeed is that $|dk_{t+1}/dk_t| < \frac{1}{2}$ and $|dk_{t+1}/dk_{t-1}| < \frac{1}{2}$, where the values dk_{t+1}/dk_t and dk_{t+1}/dk_{t-1} are obtained by applying the implicit function theorem to (20.G.5) and evaluating at the steady state. Indeed, if this condition holds, then the initial perturbation of k_1 induces a sequence of adjustments that dampen over time and that will, therefore, never become unfeasible (and, in fact, will remain bounded and strictly interior). Explicitly:

$$\frac{dk_{t+1}}{dk_t} = -\frac{\nabla^2_{22}u(\cdot) + \nabla^2_{23}u(\cdot) + \delta \nabla^2_{11}u(\cdot)}{\delta(\nabla^2_{12}u(\cdot) + \nabla^2_{13}u(\cdot))}. \tag{20.G.6}$$

If there are no externalities [i.e., if $\nabla^2_{23}u(\cdot) = \nabla^2_{13}u(\cdot) = 0$] then the concavity of $u(\cdot, \cdot)$ implies that expression (20.G.6) is larger than 1 in absolute value (you should verify this in Exercise 20.G.5). Thus, in agreement with the discussion of Section 20.D, we are not then able to find a non-steady-state solution of the Euler equations. But if the externality effects are significant enough, inspection of expression (20.G.6) tells us immediately that dk_{t+1}/dk_t can perfectly well be less than $\frac{1}{2}$ in absolute value. The same is true for $dk_{t+1}/\delta k_{t-1}$, and therefore we can conclude that robust examples with a continuum of equilibria are possible.

20.H Overlapping Generations

In the previous sections we have studied economies that, formally, have an overlapping structure of firms but only one (or, in Section 20.G, several), infinitely long-lived, consumer. We pointed out in Section 20.B that in the presence of suitable forms of altruism it may be possible to interpret an infinitely long-lived agent as a dynasty. We will now describe a model where this cannot be done, and where, as a consequence, the consumption side of the economy consists of an infinite succession of consumers in an essential manner. To make things interesting, these consumers, to be called *generations*, will overlap, so that intergenerational trade is possible. The model originates in Allais (1947) and Samuelson (1958) and has become a workhorse of macroeconomics, monetary theory, and public finance. The literature on it is very extensive; see Geanakoplos (1987) or Woodford (1984) for an overview. Here we will limit ourselves to discussing a simple case with the purpose of highlighting, first, the extent to which the model can be analyzed with the Walrasian equilibrium methodology and, second, the departures from the broad lessons of the previous sections. We shall classify these departures into two categories: issues relating to optimality and issues relating to the multiplicity of equilibria.

We begin by describing an economy that, except for the infinity of generations, is as simple as possible. We have an infinite succession of dates $t = 0, 1, \dots$ and in every period a single consumption good. For every t there is a generation born at time t, living for two periods, and having utility function $u(c_{bt}, c_{at})$ where c_{bt} and c_{at} are, respectively, the consumption of the tth generation when young (i.e., in period t), and its consumption when old (i.e., in period $t + 1$); the indices b and a are mnemonic symbols for "before" and "after." Note that the utility functions of the different generations over consumption in their lifespan are identical. We assume that $u(\cdot, \cdot)$ is quasiconcave, differentiable and strictly increasing.

Every generation t is endowed when young with a unit of a primary factor (e.g., labor). This primary factor does not enter the utility function and can be used to produce consumption goods contemporaneously by means of some production function $f(z)$.[33] Say that $f(1) = 1$. Under the competitive price-taking assumption, total profits at t, in terms of period-t good, will be $\varepsilon = 1 - f'(1)$ and, correspondingly, labor payments will be $1 - \varepsilon$. Thus, we may as well directly assume that the initial endowments of generation $t \geq 0$ are specified to us as a vector of consumption goods $(1 - \varepsilon, 0)$. In addition, we assign the infinite stream of profits to generation 0. That is, the technology $f(\cdot)$ is an infinitely long-lived asset owned at $t = 0$ by the only generation alive in that period and yielding a permanent profit stream of $\varepsilon > 0$ units of consumption good.

Now let $(p_0, \dots, p_t, \dots)$ be an infinite sequence of (anticipated) prices. We do not require that it be bounded. For the budget constraint of the different generations we take

$$p_t c_{bt} + p_{t+1} c_{at} \leq (1 - \varepsilon) p_t \qquad \text{for } t > 0 \tag{20.H.1}$$

and

$$p_0 c_{b0} + p_1 c_{a0} \leq (1 - \varepsilon) p_0 + \varepsilon \left(\sum_t p_t \right) + M. \tag{20.H.2}$$

These budget constraints deserve comment. For $t > 0$, (20.H.1) is easy to interpret. The value of the initial endowments, available at t, is $(1 - \varepsilon) p_t$. Part of this amount is spent at time t and the rest, $(1 - \varepsilon) p_t - p_t c_{bt}$, is saved for consumption at $t + 1$. The saving instrument could be the title to the technology, which would thus be bought from the old by the young at t and then sold at $t + 1$ to the new young (after collecting the period $t + 1$ return). The price paid for the asset is the amount saved, that is, $(1 - \varepsilon) p_t - p_t c_{bt}$. The direct return at $t + 1$ is εp_{t+1} and so, if the asset market is to be in equilibrium, the selling price at $t + 1$ should be $(1 - \varepsilon) p_t - p_t c_{bt} - \varepsilon p_{t+1}$. In summary, in agreement with the budget constraint (20.H.1) this leaves $(1 - \varepsilon) p_t - p_t c_{bt}$ to be spent at $t + 1$.

The constraint (20.H.2) for $t = 0$ is more interesting. Its right-hand side is the value of the asset to generation 0. Note that asset market equilibrium requires that

33. The assumption that production is contemporary with input usage fits well with the length of the period being long.

this value should be at least the *fundamental* value, that is, $\varepsilon(\sum_t p_t)$.[34] Indeed, the value of the asset at $t = 0$ equals the profit return εp_0 plus the price paid by the young of generation 1. At any T, the price paid by the young of generation T should not be inferior to the direct return εp_{T+1}. In turn, at $T - 1$ it should not be inferior to the direct return plus the value at T; that is, it should be at least $\varepsilon(p_T + p_{T+1})$. Iterating, we get the lower bound $\varepsilon(p_1 + \cdots + p_{T+1})$ for the price paid by generation 1, which, going to the limit and adding εp_0, gives $\varepsilon(\sum_t p_t)$ as a lower bound for the value to generation 0. Thus, in terms of expression (20.H.2) a necessary condition for equilibrium is $M \geq 0$. In principle, however, we should allow for the possibility of a *bubble* in the value of the asset (i.e., of $M > 0$). We did not do so in Sections 20.D or 20.G because with a *finite* number of consumers, bubbles are impossible at equilibrium. The equality of demand and supply implies that the (finite) value of total endowments plus total profits equals the value of total consumption, and therefore at equilibrium no individual value of consumption can be larger than the corresponding individual value of endowments and profit wealth (you should verify this in Exercise 20.H.1). We will see shortly that under some circumstances bubbles can occur at equilibrium with infinitely many consumers. It would therefore not be legitimate to eliminate them by definition.

The definition of a *Walrasian equilibrium* is now the natural one presented in Definition 20.H.1.

Definition 20.H.1: A sequence of prices $(p_0, \ldots, p_t, \ldots)$, an $M \geq 0$, and a family of consumptions $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$ constitute a *Walrasian* (or *competitive*) *equilibrium* if:

(i) Every (c^*_{bt}, c^*_{at}) solves the individual utility maximization problem subject to the budget constraints (20.H.1) and (20.H.2).

(ii) The feasibility requirement $(c^*_{a,t-1} + c^*_{bt} = 1)$ is satisfied for all $t \geq 0$ (we put $c^*_{a,-1} = 0$).

In a process reminiscent of the iterative procedure (presented in Section 20.D) for the determination of the policy function from the Euler equations, Figures 20.H.1 and 20.H.2 describe how we could attempt to construct an equilibrium. Normalize to $p_0 = 1$. Suppose that we now try to arbitrarily fix c_{a0}. At equilibrium, $c_{b0} = 1$, and thus p_1 is determined by the fact that p_1/p_0 must equal the marginal rate of substitution of $u(\cdot, \cdot)$ at $(1, c_{a0})$. Also, $c_{b1} = 1 - c_{a0}$. This now determines p_2. Indeed, p_2, the price at period 2, should be fixed at a value that induces a level of demand by generation 1 in period 1 of precisely c_{b1} [under the budget set given by p_1, p_2 and wealth $(1 - \varepsilon)p_1$]. With this, the demand of generation 1 in period 2, and therefore the residual amount c_{b2} left in that period for generation 2, has also been determined. But then we may be able to fix p_3 at a value that precisely induces the right amount of demand by generation 2 in period 2, that is, c_{b2}. If we can pursue this construction indefinitely so as to generate an infinite sequence $(p_1, \ldots, p_t, \ldots)$, then we have found an equilibrium. In Figure 20.H.1, where $\varepsilon > 0$, there is a single price sequence (with $p_0 = 1$) that can be continued indefinitely, and therefore a single equilibrium path.

34. Strictly speaking, we are saying that if the consumption good prices are given by $(p_0, \ldots, p_t, \ldots)$ *and* the asset prices present no arbitrage opportunity, then the price of the asset should be at least as large as its fundamental value.

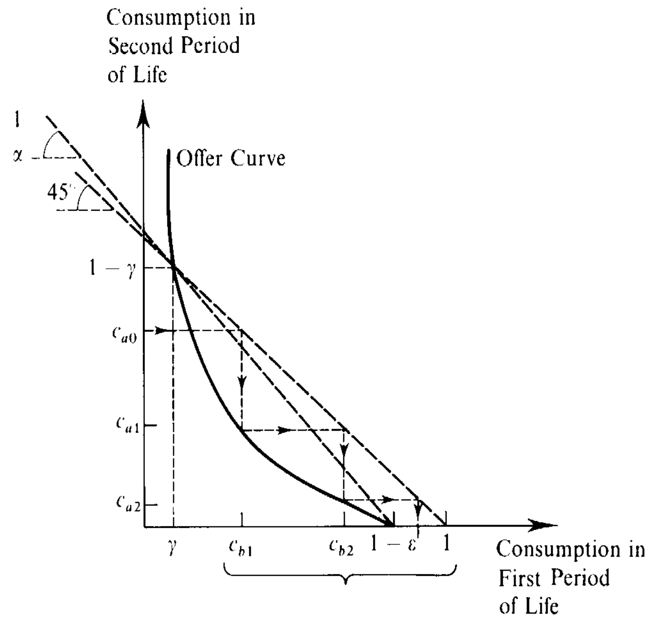


Figure 20.H.1 Overlapping generations: construction of the equilibrium (case $\varepsilon > 0$).

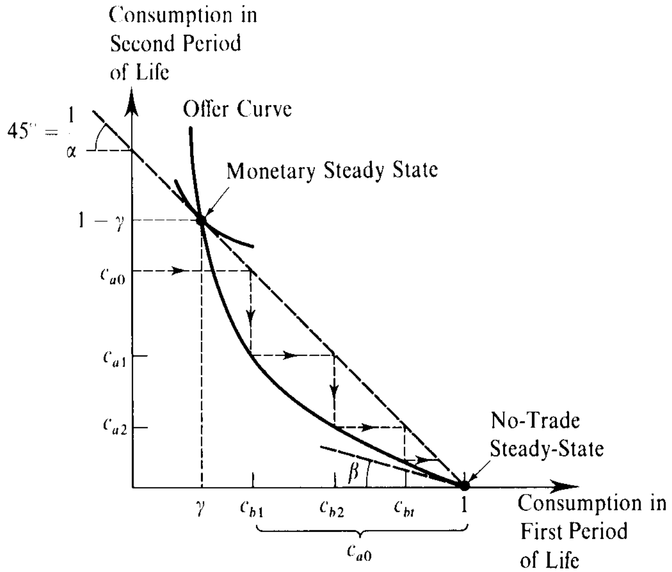


Figure 20.H.2 Overlapping generations: construction of equilibria (case $\varepsilon = 0$).

It corresponds to the stationary consumptions $(\gamma, 1 - \gamma)$ and the price sequence $p_t = \alpha^t$, where $\alpha = (1 - \varepsilon - \gamma)/(1 - \gamma) < 1$. Note that the iterates that begin at a value $c_{a0} \neq 1 - \gamma$ unavoidably "leave the picture," that is, become unfeasible. In Figure 20.H.2, where $\varepsilon = 0$, there is a continuum of equilibria: any initial condition $c_{a0} \leq 1 - \gamma$ can be continued indefinitely.

It is plausible from Figures 20.H.1 and 20.H.2 that the existence of an equilibrium can be guaranteed under general conditions. This is indeed the case [see Wilson (1981)].

Pareto Optimality

Suppose first that $\varepsilon > 0$. We say then that the asset is *real* (it has "real" returns). At an equilibrium the wealth of generation 0, $(1 - \varepsilon)p_0 + \varepsilon(\sum_t p_t) + M$, must be finite (how could this generation be in equilibrium otherwise?). Therefore, if $\varepsilon > 0$, it follows that $\sum_t p_t < \infty$.[35] An important implication of this is that the *aggregate* (i.e., added over all generations) *wealth of society*, which is precisely $\sum_t p_t$, *is finite*. In Proposition 20.H.1 we now show that, as a consequence, the first welfare theorem applies for the model with $\varepsilon > 0$.

Proposition 20.H.1: Any Walrasian equilibrium $(p_0, \ldots, p_t, \ldots)$, $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$, with $\sum_t p_t < \infty$ is a Pareto optimum; that is, there are no other feasible consumptions $\{(c_{bt}, c_{at})\}_{t=0}^{\infty}$ such that $u(c_{bt}, c_{at}) \geq u(c^*_{bt}, c^*_{at})$ for all $t \geq 0$, with strict inequality for some t.

Proof: We repeat the standard argument. Suppose that $\{(c_{bt}, c_{at})\}_{t=0}^{\infty}$ Pareto dominates $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$. From feasibility, we have $c^*_{bt} + c^*_{a,t-1} = 1$ and $c_{bt} + c_{a,t-1} \leq 1$ for every t. Therefore, $\sum_t p_t(c^*_{bt} + c^*_{a,t-1}) = \sum_t p_t$ and $\sum_t p_t(c_{bt} + c_{a,t-1}) \leq \sum_t p_t$. Because $\sum_t p_t < \infty$, we can rearrange terms and get

$$\sum_t (p_t c_{bt} + p_{t+1} c_{at}) \leq \sum_t (p_t c^*_{bt} + p_{t+1} c^*_{at}) = \sum_t p_t < \infty.$$

Because the utility function is increasing and (c^*_{bt}, c^*_{at}) maximizes utility in the budget set we conclude that $p_t c_{bt} + p_{t+1} c_{at} \geq p_t c^*_{bt} + p_{t+1} c^*_{at}$ for every t, with at least one strict inequality. Therefore, $\sum_t (p_t, c_{bt} + p_{t+1} c_{at}) > \sum_t (p_t c^*_{bt} + p_{t+1} c^*_{at})$. Contradiction. ■

Proposition 20.H.1 is important but it is not the end of the story. Suppose now that the asset is purely *nominal* (i.e., $\varepsilon = 0$; for example, the asset could be fiat money, or ownership of a constant returns technology). Then *it is possible to have equilibria that are not optimal.* In fact, it is easy to see that we can sustain autarchy (i.e., no trade) as an equilibrium. Just put $M = 0$ (no bubble, worthless fiat money) and choose $(p_0, \ldots, p_t, \ldots)$ so that, for every t, the relative prices p_t/p_{t+1} equal the marginal rate of substitution of $u(\cdot, \cdot)$ at $(1, 0)$, denoted by β. This no-trade stationary state (also called *the nonmonetary steady state*) where every generation consumes $(1, 0)$ is represented in Figure 20.H.2. As it is drawn (with $\beta < 1$), we can also see that the no-trade outcome is strictly Pareto dominated by the steady state $(\gamma, 1 - \gamma)$ [or, more precisely, by the consumption path in which generation 0 consumes $(1, 1 - \gamma)$ and every other generation consumes $(\gamma, 1 - \gamma)$]. What is going on is simple: in this example the open-endedness of the horizon makes it possible for the members of every generation t to pass an extra amount of good to the older generation at t and, at the same time, be more then compensated by the amount passed to them at $t + 1$ by the next generation. Note that, in agreement with Proposition 20.H.1, the lack of optimality of this no-trade equilibrium entails $p_t/p_{t+1} = \beta < 1$ for all t; that is, prices increase through time.

It is also possible in the purely nominal case for an equilibrium with $M > 0$ not to be Pareto optimal. Note first if $\{(c^*_{bt}, c^*_{at})\}_{t=0}^{\infty}$, $(p_0, \ldots, p_t, \ldots)$ and M constitute an

35. You can also verify this graphically by examining Figure 20.H.1.

equilibrium, then we have (recall that $c^*_{b0} = 1$)

$$p_{t+1}c^*_{at} = p_t(1 - c^*_{bt}) = p_t c^*_{a,t-1} = \cdots = p_1 c^*_{a0} = M \qquad \text{for every } t.$$

Thus, $M = 0$ can occur only at a no-trade equilibrium. In Figure 20.H.2, there is a continuum of equilibria indexed by c_{b1} for $\gamma \le c_{b1} \le 1$. The no-trade equilibrium corresponds to $c_{b1} = 1$. But for every $c_{b1} < 1$ with $c_{b1} > \gamma$ we have a nonstationary equilibrium trajectory with trade (hence $M > 0$) which is also strictly Pareto dominated by the steady state $(\gamma, 1 - \gamma)$. Nonetheless, it is still true that for any equilibrium with $c_{b1} > \gamma$ we have $M/p_t \to 0$; that is, in real terms the value of the asset becomes vanishingly small with time. For $c_{b1} = \gamma$, matters are quite different. We have a steady-state equilibrium (called *the monetary steady state*) in which the price sequence p_t is constant and therefore the real value of money remains constant and positive. This monetary steady state is the analog of the *golden rule* of Section 20.E and, as was the case there, we have that, in spite of $\sum_t p_t < \infty$ being violated, the *monetary steady state is Pareto optimal.* We will not give a rigorous proof of this. The basic argument is contained in Figure 20.H.3. There we represent the indifference curve through $(\gamma, 1 - \gamma)$ and check that any attempt at increasing the utility of generation 0 by putting $c_{b1} < \gamma$ leads to an unfeasible chain of compensations; that is, it cannot be done.

The discussion just carried out of the examples in Figures 20.H.2 and 20.H.3 suggests and confirms the following claim, which we leave without proof: *In the purely nominal case, of all equilibrium paths the Pareto optimal ones are those, and only those, that exhibit a bubble whose real value is bounded away from zero throughout time.*

It is certainly interesting that a bubble can serve the function of guaranteeing the optimality of the equilibria of an economy, but one should keep in mind that this happens only because an asset is needed to transfer wealth through time. If a real asset exists then this asset can do the job. If one does not exist then the economy, so to speak, needs to invent an asset. To close the circle, we point out that if there is a real asset then not only is a bubble not needed but, in fact, it cannot occur.

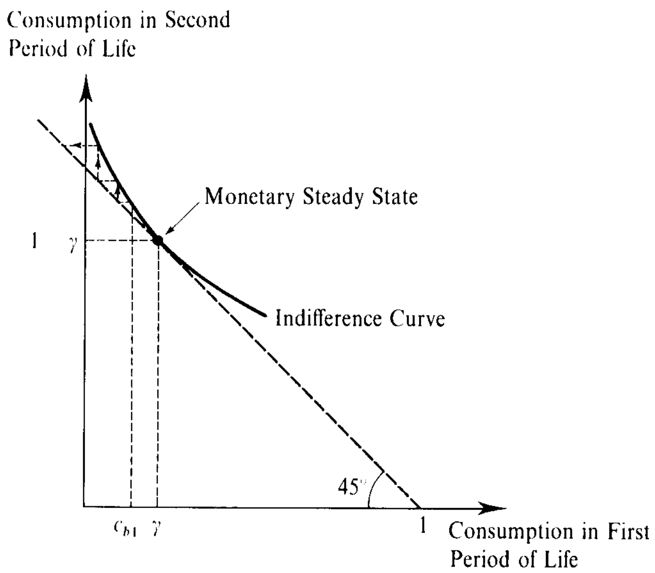


Figure 20.H.3
The monetary steady state is Pareto optimal.

Proposition 20.H.2: Suppose that at an equilibrium we have $\sum_t p_t < \infty$. Then $M = 0$.

Proof: The sum of wealths over generations is $\sum_t p_t + M < \infty$. The value of total consumption is $\sum_t p_t < \infty$. The second amount cannot be smaller than the first (otherwise some generation is not spending its entire wealth). Therefore $M = 0$. ■

Multiplicity of Equilibria

We have already seen, in Figure 20.H.2, a model with a purely nominal asset (i.e., $\varepsilon = 0$) and very nicely shaped preferences (the offer curve is of the gross substitute type) for which there is a continuum of equilibria. Of those, one is the Pareto optimal monetary steady state and the rest are nonoptimal equilibria where the real value of money goes to zero asymptotically. The existence of this sort of indeterminacy is clearly related to the ability to fix with some arbitrariness the real value of money (the "bubble") at $t = 0$, that is M/p_0. It cannot occur if bubbles are impossible, as, for example, in the model with a real asset (i.e., $\varepsilon > 0$) where, in addition, we know that the equilibrium is Pareto optimal.

One may be led by the above observation to suspect that the failure of Pareto optimality is a precondition for the presence of a robust indeterminacy (i.e., of a continuum of equilibria not associated with any obvious coincidence in the basic data of the economy). This suspicion may be reinforced by the discussion of Section 20.G, where we saw that the Pareto optimality of equilibria was key to our ability to claim the generic determinacy of equilibria in models with a finite number of consumers. Unfortunately, with overlapping generations the number of consumers is infinite in a fundamental way,[36] and this complicates matters. Whereas with a real asset the Pareto optimality of equilibria is guaranteed and the type of indeterminacy of Figure 20.H.2 disappears, it is nevertheless possible to construct nonpathological examples with a continuum of equilibria.

The simplest example is illustrated in Figure 20.H.4. The figure describes a real-asset model with the steady state $(\gamma, 1 - \gamma)$. Suppose that, in a procedure we have resorted to repeatedly, we tried to construct an equilibrium with c_{a0} slightly different from $1 - \gamma$. Then, normalizing to $p_0 = 1$, we would need to use p_1 to clear the market *of period* 0, p_2 to do the same for period 1, and so on. In the leading case of Figure 20.H.1, we have seen that this eventually becomes unfeasible. A change in p_t that takes care of a disequilibrium at $t - 1$ creates an even larger disequilibrium at t, which then has to be compensated by a change of a larger magnitude in p_{t+1} in an explosive process that finally becomes impossible. But in Figure 20.H.4, the utility function is such that, at the relative prices of the steady state, a change in the price of the second-period good has a larger impact on the demand for the first-period good than on the demand for the second-period good. Hence, the successive adjustments necessitated by an initial disturbance from $c_{a0} = 1 - \gamma$ dampen with each iteration and can be pursued indefinitely. We conclude that an equilibrium exists with the new initial condition. As a matter of terminology, the

36. By this vague statement we mean that there is no way we could assert that the infinitely many consumers are sufficiently similar for them to be "approximated" by a finite number of representatives.

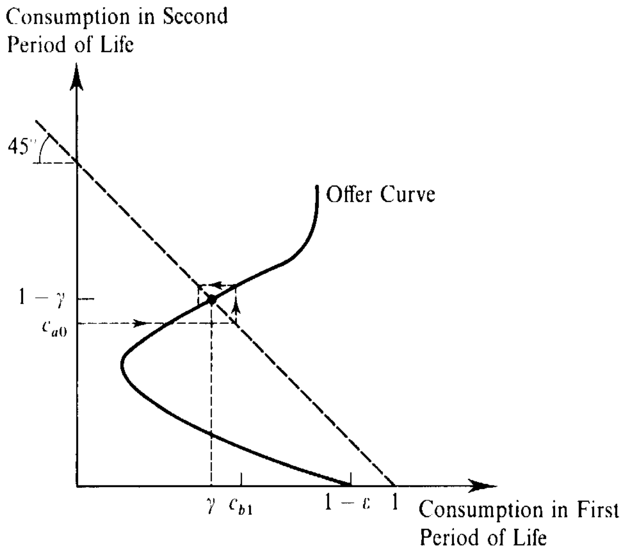


Figure 20.H.4
An example of a continuum of (Pareto optimal) equilibria in the real asset case.

locally isolated steady state equilibrium of Figure 20.H.1 is called *determinate*, and the one of Figure 20.H.4 is called *indeterminate*.[37]

It is interesting to point out that the leading case of unique equilibrium (Figure 20.H.1) in a real-asset model corresponds to a gross substitute excess demand function, while Figure 20.H.4 represents the sort of pronounced complementaries that were sources of nonuniqueness in the examples of Sections 15.B (recall also the discussion of uniqueness in Section 17.F). The connection between nonuniqueness and indeterminateness is actually quite close, and you are asked to explore it in Exercise 20.H.2. Here we simply mention that gross substitution is not a necessary condition for uniqueness. It can be checked, for example, that in the real asset model the steady state remains the only equilibrium if consumption in both periods is normal in the demand function of $u(\cdot, \cdot)$ and if the corresponding excess demand $(z_b(p_b, p_a), z_a(p_b, p_a))$ satisfies

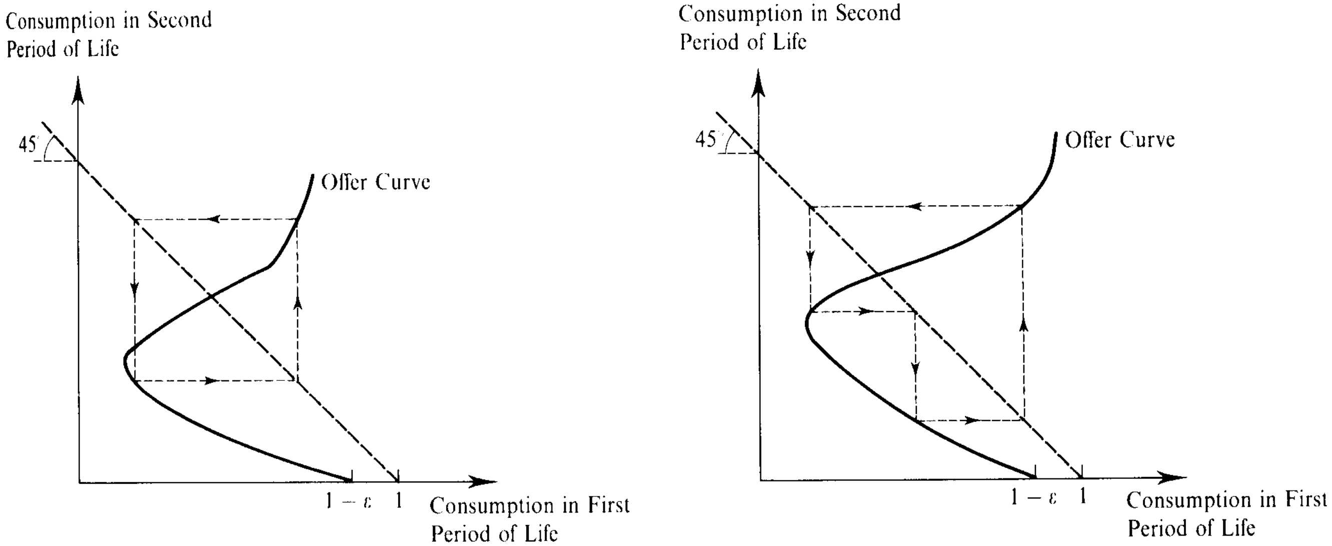
$$\nabla_1 z_b(p_b, p_a) < \nabla_1 z_a(p_b, p_a) \qquad \text{for all } p_b, p_a. \tag{20.H.3}$$

Expression (20.H.3) permits a price increase in the first period of life to lead to an increase in demand in this period (a possibility ruled out by gross substitution); but, if so, it requires the increase of demand in the second period of life to be larger. Geometrically speaking, the condition is that the slope of the offer curve in the (c_b, c_a) plane should never be positive and less than 1. Note that in Figure 20.H.4 this is violated at the steady state. Condition (20.H.3) is known as the *determinacy condition*. If the reverse inequality holds at the steady state, then, as in Figure 20.H.4, there is a continuum of equilibria all converging to the steady state (the steady state is therefore *indeterminate*).

37. Observe that, at least in the context of the relatively simple model we are now discussing, there is little room for cases intermediate between uniqueness or the existence of a continuum of equilibria.

In Chapter 17 (see Section 17.D and Appendix A of Chapter 17) we argued that, with Pareto optimality, an equilibrium problem with a finite number of consumers could be represented by means of a finite number of equations with the same number of unknowns. From this we claimed that generic determinacy was the logical conjecture to make for this case. In Section 20.G we extended this argument to the model with a finite number of infinitely long-lived consumers. However, the current overlapping generations problem has a basic difference in formal structure: there is no natural trick allowing us to see the equilibrium as anything but the zeros of an infinite system of equations (of the excess demand type, say). Mathematically, this is significant. To give an example, intimately related to the issues we are discussing, suppose that $f: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map that is onto (i.e., $f(x) = Ax$, where A is a nonsingular matrix). Then 0 is the unique solution to $f(x) = 0$. But suppose now that $f(\cdot)$ maps bounded sequences into bounded sequences and that it is linear and onto. Then $f(x) = 0$, or, equivalently, $f_t(x_1, \ldots, x_t, \ldots) = 0$ for all t, need not have a unique solution. A simple example is the backward shift, that is, $f_t(x_1, \ldots, x_t, \ldots) = x_{t+1}$, where any $(\alpha, 0, \ldots, 0, \ldots)$ is sent to zero.

What can we say about the dynamics of an equilibrium? We saw that the "anything goes" principle applied to the one-consumer model. It would be surprising if it did not apply here; indeed, in Figures 20.H.5 and 20.H.6 we provide nonpathological examples with cycles.[38] Note



that in Figure 20.H.6 we have a three-period cycle: chaos rears its head. In the gross substitute example of Figure 20.H.1 we have monotone convergence to the steady-state. In a sense, the gross substitute case is the analog of the approach based on the sign of the second derivatives described in Section 20.F. Note that in the overlapping generations situation the factor of discount is not a meaningful concept and, therefore, there is no analog of a dynamic theory based on patience. In Section 20.G we also mentioned, quite loosely, that there did not seem to be, for the case of a finite number of agents with Pareto optimality, a close relation between the determinacy and the dynamic properties of equilibrium. In the current setting the connection is closer, at least in the following sense: If equilibrium trajectories with cycles can occur, then there are infinitely many equilibria.

Figure 20.H.5 (left) Complementary consumptions: example of a period-2 equilibrium path.

Figure 20.H.6 (right) Complementary consumptions: example of a period-3 equilibrium path.

38. In particular, no inferior goods are required for these examples.

20.I Remarks on Nonequilibrium Dynamics: Tâtonnement and Learning

The dynamic analysis that has concerned us so far in this chapter is of a different nature from, and should not be confused with, the dynamics studied in Section 17.H. The dynamics here display the temporal unfolding of an equilibrium (an internal property of the equilibrium, in the terminology of Section 20.G), whereas in Section 17.H we were trying to assess the dynamic forces that, in real or in fictional time, would buffet an economy disturbed from its equilibrium (hence, we were looking at an external property). As we saw, nonequilibrium dynamic analysis raises a host of conceptual problems, yet it may offer useful insight into the plausibility of the occurrence of particular equilibria. This remains valid in the setting of intertemporal equilibrium.

Abstracting from technical complexities, the analysis and the results of Section 17.H can be adapted and hold true for the infinite-horizon, finite number of consumers model of Section 20.G. On the other hand, as we have seen, the temporal framework has its own special theory, which could conceivably be illuminated by specific nonequilibrium considerations. We make three remarks in this direction.

Short-Run Equilibrium and Permanent Income[39]

Suppose that $(p_0, \ldots, p_t, \ldots)$ is the equilibrium price sequence of an economy with L goods and I consumers. Consumers are as in Section 20.D. Then at the equilibrium consumptions we have (assuming interiority)

$$\delta^t \nabla u_i(c_{ti}) = \lambda_i p_t \qquad \text{for all } t \text{ and every } i. \tag{20.I.1}$$

This is just (20.D.6). The variable λ_i is the marginal utility of income, or wealth, and the vector of reciprocals $(\eta_1, \ldots, \eta_I) = (1/\lambda_1, \ldots, 1/\lambda_I)$ can serve as the weights for which the given equilibrium maximizes the weighted sum of utilities (see Section 20.G).

It follows from (20.I.1) that the short-run demands (i.e., the demands at $t = 0$) are entirely determined by p_0 and the marginal utilities of wealth λ_i. Denote this demand by $c_{0i}(p, \lambda_i)$. In the spirit of tâtonnement dynamics, suppose that p_0 is perturbed to some p_0'. What will happen to demand at $t = 0$? If the λ_i remain fixed, then (20.I.1) implies that short-run demand behaves as the demand for non-numeraire goods in a quasilinear utility model with concave utility functions. In particular, differentiating (20.I.1) we see that the $L \times L$ matrix of short-run price effects

$$D_{p_0} c_{0i}(p_0, \lambda_i) = \lambda_i [D^2 u_i(c_{0i})]^{-1}$$

is negative definite (by the concavity of $u_i(\cdot)$) and, therefore, so is the aggregate $\sum_i D_{p_0} c_{0i}(p_0, \lambda_i)$. In more economic terms, as long as the λ_i stay fixed there are no wealth effects present in the short-run demands. Substitution prevails and, consequently, the short-run equilibrium is unique and globally tâtonnement stable.

In reality, however, after a change in p_0 we should expect that λ_i will have changed at the new consumer optimum. But if the rate of discount is close to 1 (i.e., if agents

39. See Bewley (1977) for more on this topic. The term "permanent income" is standard and so we use it rather than "permanent wealth."

are patient) then the change in λ_i should be small: The current period is not significantly more important than any other period and, therefore, it will account for only a small fraction of total utility and expenditure. Hence, we could say that partial equilibrium analysis is justified in the short run (recall the discussion of partial equilibrium analysis in Section 10.G). In summary: *If consumers are sufficiently patient, then the short-run equilibrium is unique and globally stable (for the tâtonnement dynamics).*

The (Short-Run) Law of Demand in Overlapping Generations Models

We now look at the short-run equilibrium of the overlapping generations model of Section 20.H. This is an example of a model where wealth effects matter in the short run and, therefore, the permanent income approach does not apply. We consider the version of the model with a real asset and normal goods and ask whether the stability of the fictional-time tâtonnement dynamics at a given date t helps us to distinguish among types of equilibria. Because there is a single good per period, the stability criterion for a single period is simple enough—it amounts to the law of demand at time t. That is, we say that an equilibrium $(p_0, \ldots, p_t, \ldots)$ is tâtonnement stable at time t if an (anticipated) increase in p_t, all other prices remaining fixed, results in excess supply in that period (note that only generations $t-1$ and t will alter their consumption plans).

We know that if the excess demand function of the generations is of the gross substitute type, then there is a unique equilibrium (which is in steady state). (See Figure 20.H.1.) Moreover, the definition of gross substitution tells us that the law of demand is satisfied at any t. This gives us a first link between the notions of determinate equilibrium and tâtonnement stability. This link can be pushed beyond the gross substitute case. Take a steady-state equilibrium price sequence $(1, \rho, \ldots, \rho^t, \ldots)$. By the homogeneity of degree zero of excess demand functions $(z_a(\cdot,\cdot), z_b(\cdot,\cdot))$, which implies the homogeneity of degree -1 of $\nabla z_a(\cdot,\cdot)$ and $\nabla z_b(\cdot,\cdot)$, we have (you should verify this in Exercise 20.I.1)

$$\nabla_2 z_a(1/\rho, 1) + \nabla_1 z_b(1, \rho) = \rho \nabla_2 z_a(1, \rho) + \nabla_1 z_b(1, \rho) = -\nabla_1 z_a(1, \rho) + \nabla_1 z_b(1, \rho).$$

The negativity of the left-hand side is the tâtonnement stability criterion, that is, the law of demand at a single market,[40] while the negativity of the right-hand side (i.e., the requirement that wealth effects are not so askew that a decrease in the price in one period increases the demand of the young in that period by less than it increases the demand of these same young for their consumption in the next period) is the criterion for the determinacy of the steady state [see expression (20.H.3)]. Recall that determinate means that there is no other equilibrium trajectory that remains in an arbitrarily small neighborhood of the steady state. We conclude that there is an exact correspondence: *a steady-state equilibrium is (short-run, locally) tâtonnement stable at any t if and only if it is determinate.*[41]

40. If p_t is changed infinitesimally then the demand of the old changes by $\nabla_2 z_a(\rho^{t-1}, \rho^t)$ while the demand of the young changes by $\nabla_1 z_b(\rho^t, \rho^{t+1})$. Because $\nabla_2 z(\cdot,\cdot)$ and $\nabla_1 z(\cdot,\cdot)$ are homogeneous of degree -1, the total change equals $(1/\rho^t)\,\nabla_2 z_a(1/\rho, 1) + (1/\rho^t)\,\nabla_1 z_b(1, \rho)$

41. In this "if and only if" statement we neglect borderline cases.

We have confined ourselves to the real asset case to avoid a complication. With a purely nominal asset the previous concept of tâtonnement stability loses the power to discriminate between determinate and indeterminate steady-state equilibria, unless we restrict ourselves a priori to monetary steady states (to see this, consider the simplest gross substitute case). The learning concept to be presented in the remainder of this section does not suffer from this limitation.

Learning

We now briefly discuss a nonequilibrium dynamics that takes place in real time and that can be interpreted in terms of learning. The framework is that of the overlapping generations of Section 20.H and, to be as simple as possible, we focus on the purely nominal asset case.

We describe first how the short-run equilibrium (i.e., the equilibrium at a given period t) is determined. We suppose that there is a certain fixed amount of fiat money M (denominated, say, in dollars). The excess demand of the older generation at date $t \geq 1$ is then M/p_t. The excess demand of the younger generation at the same date depends on p_t but also on the *expectation* p^e_{t+1} of the price at $t+1$. Given p^e_{t+1}, the price p_t is a *temporary equilibrium at time* $t \geq 1$ if $z_b(p_t, p^e_{t+1}) + (M/p_t) = 0$. Thus, given a sequence of price expectations $(p^e_1, \dots, p^e_t, \dots)$, we generate a sequence of temporary equilibrium prices $(p_1, \dots, p_t, \dots)$.

But, how are expected prices determined? To take them as given does not make much sense. The sequence of realizations should feed back into the sequence of expectations. The self-fulfilled, or rational, expectations approach (which we have implicitly adhered to in this chapter) imposes a correct expectations condition: $p^e_{t+1} = p_{t+1}$ for every t.[42] An alternative is to require that p^e_{t+1} (the price expected at t to prevail at $t+1$) be an extrapolation of the past (and current) realizations $p_0, \dots, p_t$. In this approach we think of consumers as engaged in some sort of learning and of expectations responding *adaptively* to experienced outcomes.[43]

To be specific, let us take a not very realistic, but very simple, extrapolation rule: $p^e_{t+1} = p_{t-1}$ (i.e., the price at $t+1$ expected by the young at $t \geq 1$ is the price that ruled in the most recent past). Equivalently (given the fixed amount of fiat money M), the young at t expect to consume at $t+1$, when old, the same amount consumed by the old at $t-1$. The equation for the determination of p_t is then

$$z_b(p_t, p_{t-1}) = -\frac{M}{p_t}. \tag{20.I.2}$$

By Walras' law, (20.I.2) can equivalently be written as

$$z_a(p_t, p_{t-1}) = \frac{M}{p_{t-1}}. \tag{20.I.3}$$

42. The term "self-fulfilled" is justified because the sequence of expectations $(p^e_1, \dots, p^e_t, \dots)$ induces a sequence of realizations identical to itself. The term "rational" is justified by the fact that, given $(p^e_1, \dots, p^e_t, \dots)$, a member of generation t should, in principle, be able to compute the price realization p_{t+1} and therefore verify the correctness of p^e_{t+1}.

43. We should emphasize, first, that all this is a nonequilibrium story and, second, that we cannot rigorously discuss learning without explicitly introducing an uncertain environment.

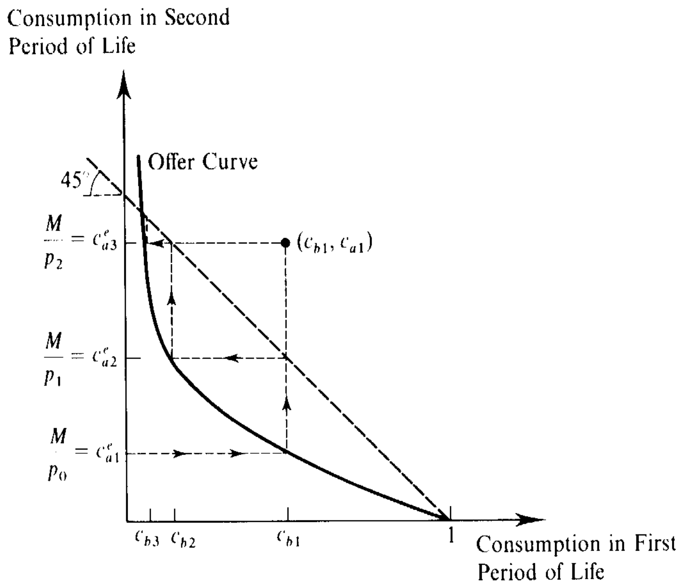


Figure 20.I.1
Learning dynamics.

Given an arbitrary initial condition p_0, we can then compute the sequence of temporary equilibrium realizations $(p_1, \ldots, p_t, \ldots)$ by iteratively using (20.I.2) or (20.I.3). Note that in doing so, the planned excess demands in (20.I.2) will be realized but those in (20.I.3) may not (because p_{t+1} may not be equal to p_{t-1}). We represent the dynamic process in Figure 20.I.1. In the figure, c_{bt} and c^e_{at} stand for the planned consumptions of generation t at times t and $t+1$, respectively. Given M/p_{t-1} we get c^e_{at} from (20.I.3), and c_{bt} from the fact that planned consumptions are in the offer curve. Finally (20.I.2) moves us to the next value M/p_t. For generation 1 we also show the actual consumption vector (c_{b1}, c_{a1}).

From Figure 20.I.1 we can see an interesting fact: The learning dynamics exactly reverses the equilibrium dynamics (compare with Figure 20.H.2).[44] For the gross substitute case shown in the figure, this means that all the trajectories tend to the monetary steady state. Hence, in the limit we have a true self-fulfilled expectations equilibrium. Consumers have learned their way into equilibrium, so to speak. For the crude learning dynamics we are considering, this need not be so for the case of a general offer curve (an infinite sequence with systematic prediction error is quite possible), but the property of exact reversal of equilibrium dynamics suffices to provide, once again, a test for the well-behavedness of steady states that reinforces the intuitions developed earlier: *A steady state is (locally) stable for the learning dynamics if and only if it is determinate (i.e., "locally isolated").*

44. More precisely, if $(p_1, \ldots, p_t, \ldots)$ is the sequence of realizations of the adaptive expectations dynamics, then for any T there is an equilibrium sequence $(p'_0, \ldots, p'_t, \ldots)$ such that $p'_t = p_{T-t}$ for every $t < T$.

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EXERCISES

20.B.1[B] Adopting the definition of *time impatience* given in comment (1) of Section 20.B, show that a utility function of the form (20.B.1) exhibits time impatience.

20.B.2[B] Verify that a utility function of the form (20.B.1) is stationary according to the definition given in comment (2) of Section 20.B. Also, exhibit a violation of stationarity with a utility function of the form $V(c) = \sum_{t=0}^{\infty} \delta_t^t u(c_t)$.

20.B.3[B] With reference to comment (3) of Section 20.B, write $c = (c', c'')$ where $c' = (c_0, \ldots, c_t)$, $c'' = (c_{t+1}, \ldots)$. Suppose that the utility function $V(\cdot)$ is additively separable. Show that if $V(\bar{c}', c'') \geq V(\bar{c}', \hat{c}'')$ for some $\bar{c}'$, then $V(c', c'') \geq V(c', \hat{c}'')$ for all c'. Show that if $V(c', \bar{c}'') \geq V(\hat{c}', \bar{c}'')$ for some c'', then $V(c', c'') \geq V(\hat{c}', c'')$ for all c''. Interpret.

20.B.4[C] Show that in a recursive utility model with aggregator function $G(u, V) = u^\alpha + \delta V^\alpha$, $0 < \alpha < 1$, $\delta < 1$, and increasing, continuous one-period utility $u(c_t)$, the utility $V(c)$ of a bounded consumption stream is well defined. [*Hint*: Use (20.B.3) to compute the utility for consumption streams truncated at a finite horizon. Then show that a limit exists as $T \to \infty$. Finally, argue that the limit satisfies the aggregator equation.]

20.B.5[A] Show that the utility function $V(c)$ on consumption streams given by (20.B.1) is concave. Show also that the additively separable form of $V(\cdot)$ is a cardinal property.

20.C.1[A] Given the price sequence $(p_0, p_1, \ldots, p_t, \ldots)$, $p_t \in \mathbb{R}^L$, define for every t and every commodity ℓ the rate of interest from t to $t+1$ in terms of commodity ℓ (this is known as the *own rate of interest* of commodity ℓ at t).

20.C.2[A] Show that if the path $(y_0, \ldots, y_t, \ldots)$, is myopically profit maximizing for $(p_0, p_1, \ldots, p_t, \ldots) \geq 0$, then $(y_0, \ldots, y_t, \ldots)$ is also profit maximizing for $(p_0, p_1, \ldots, p_t, \ldots)$ over any finite horizon, in the sense that, for any T, the total profits over the first T periods cannot be increased by any coordinated move involving only these periods.

20.C.3[A] Define an appropriate concept of weak efficiency and reprove Proposition 20.C.1, requiring only that $(p_0, \ldots, p_t, \ldots)$ is a nonnegative sequence with some nonzero entry.

20.C.4[B] Suppose that the production path $(y_0, \ldots, y_t, \ldots)$ is bounded (i.e., there is a fixed α such that $\|y_t\| \leq \alpha$ for all t), that $(p_0, \ldots, p_t, \ldots) \gg 0$, and that $\sum_{t=0}^{\infty} p_t < \infty$. We say that the path $(y_0, \ldots, y_t, \ldots)$ is overall profit maximizing with respect to $(p_0, \ldots, p_t, \ldots)$ if

$$\sum_{t=0}^{\infty} (p_t \cdot y_{bt} + p_{t+1} \cdot y_{a,t}) \geq \sum_{t=0}^{\infty} (p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{a,t})$$

for any other production path $(y'_0, \ldots, y'_t, \ldots)$.

(a) Show that if $(y_0, \ldots, y_t, \ldots)$ is overall profit maximizing with respect to $(p_0, \ldots, p_t, \ldots) \gg 0$, then it is efficient.

(b) Show that if $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing with respect to $(p_0, \ldots, p_t, \ldots) \gg 0$, then it is also overall profit maximizing.

20.C.5[C] Say that a production path $(y_0, \ldots, y_t, \ldots)$, is T-efficient, for $T < \infty$, if there is no other production path $(y'_0, \ldots, y'_t, \ldots)$ that, first, dominates $(y_0, \ldots, y_t, \ldots)$ in the sense of efficiency and, second, is such that the cardinality of $\{t\colon y_t \neq y'_t\}$ is at most T.

(a) Show that if $(y_0, \ldots, y_t, \ldots)$ is myopically profit maximizing with respect to $(p_0, \ldots, p_t, \ldots) \gg 0$, then $(y_0, \ldots, y_t, \ldots)$ is T-efficient for all $T < \infty$.

(b) Show that if the technology is smooth (in the sense used in the small-type discussion at the end of Section 20.C; assume also that the outward unit normals to the production frontiers are strictly positive), then 2-efficiency implies T-efficiency for all $T < \infty$.

(c) (Harder) Show that the conclusion of **(b)** fails for general linear activity technologies. Exhibit an example. [*Hint*: Rely on chains of intermediate goods.]

20.C.6[A] Consider the Ramsey–Solow technology of Example 20.C.1, as continued in Example 20.C.6. The exogenous path of labor endowments is $(l_0, \ldots, l_t, \ldots)$. Given a production path $(k_0, \ldots, k_t, \ldots)$, we determine a sequence of consumption good prices $(q_0, \ldots, q_t, \ldots)$ by the requirement that $(q_t/q_{t+1}) = \nabla_1 F(k_t, l_t)$ for all t. Show then that a sequence of wages w_t can be found so that the path determined by $(k_0, \ldots, k_t, \ldots)$ is myopically profit maximizing for the price sequence determined by $((q_0, w_0), \ldots, (q_t, w_t), \ldots)$.

20.D.1[A] Consider the budget constraint of problem (20.D.3). To simplify, suppose that we are in a pure exchange situation. Write the budget constraint as a sequence of budget constraints, one for each date. To this effect, assume that money can be borrowed and lent at a zero nominal rate of interest.

20.D.2[A] Show that condition (ii′) in Section 20.D (it is stated just before Definition 20.D.2) implies condition (ii) of Definition 20.D.1. Show that, conversely, condition (ii), together with $w = \sum_t p_t \cdot \omega_t + \sum_t \pi_t < \infty$, implies condition (ii′).

20.D.3[A] in text.

20.D.4[A] Complete the computations requested in Example 20.D.1.

20.D.5[C] In the context of Example 20.D.3, compute the Euler equations for the optimal investment policy when the production function has the form $F(k) = k^\alpha$, $0 < \alpha < 1$, and the adjustment cost function is given by $g(k' - k) = (k' - k)^\beta$, with $\beta > 1$, for $k' > k$, and by $g(k' - k) = 0$ for $k' \le k$. Say as much as you can about the policy. In particular, determine the steady-state trajectory of investment.

20.D.6[B] Verify the claim made in the proof of Proposition 20.D.7 that the Euler equations (20.D.9) are the first-order necessary and sufficient conditions for short-run optimization. In other words: they are necessary and sufficient for the nonexistence of an improving trajectory differing from the given one at only a finite number of dates.

20.D.7[A] With reference to Example 20.D.4, show that, for the functional forms given, the Euler equations are as indicated in the example: $k_{t+1} = 3k_t - 2k_{t-1}$ for every t. Also verify that the solution to this difference equation given in the text is indeed a solution, that is, that it satisfies the equation.

20.D.8[A] Verify that the value function $V(k)$ does satisfy the properties (i) and (ii) claimed for it at the end of Section 20.D.

20.D.9[A] Argue that the properties (i) and (ii) of the value function referred to in Exercise 20.D.8 yield the two consequences, concerning $V'(k)$ and $V''(k)$, claimed at the end of Section 20.D.

20.E.1[A] Discuss in what sense the term r defined after the proof of Proposition 20.E.1 can be interpreted as the rate of interest implicit in the proportional price sequence.

20.E.2[B] Suppose that the production set $Y \subset \mathbb{R}^L$ is of the constant return type and consider production paths that are *proportional* (but not necessarily stationary), that is, paths $(y_0, \ldots, y_t, \ldots)$ that satisfy $y_t = (1 + n) y_{t-1}$ for all t and some n.

(a) Argue that the conclusion of Proposition 20.E.1 remains valid for proportional paths.

(b) State and prove the result parallel to Proposition 20.E.2 for proportional paths.

20.E.3[B] Suppose that in the Ramsey–Solow model $\bar{k}$ solves Max $(F(k, 1) - k)$ (see Figure 20.E.2). Show that if $k_t \le \bar{k} - \varepsilon$ for all t, then the path determined by $(k_0, \ldots, k_t, \ldots)$ is efficient. [*Hint*: Compute prices and verify the transversality condition.]

20.E.4[A] Prove the three neoclassical properties stated at the end of the regular type part of Section 20.E.

20.E.5[A] Carry out the requested verification of expression (20.E.1).

20.E.6^A Carry out the verification requested in the discussion of Figure 20.E.3.

20.E.7^A In the Ramsey Solow model, two different steady states are associated with different rates of interest. This is not so in the example illustrated in Figure 20.E.3, at first sight very similar. The key difference is that in the Ramsey–Solow model the consumption and investment goods are perfect substitutes in production. Clarify this by proving, in the context of the example underlying Figure 20.E.3, that if the two goods are perfect substitutes then $r(\bar{k}) \neq r(\bar{\bar{k}})$ whenever $\bar{k} \neq \bar{\bar{k}}$. [*Hint*: Their being perfect substitutes means that $G(k, k' + \alpha) = G(k, k') - \alpha$ for any $\alpha < F(k, k')$.]

20.E.8^A Consider the proportional production paths with rate of growth equal to $n > 0$ (recall Exercise 20.E.2) in the context of a Ramsey–Solow technology of constant returns. Show that among these paths the one that maximizes surplus (at $t = 1$, or, equivalently, normalized surplus or surplus "per capita") is characterized by having the rate of interest equal to n. This path is also called the *golden rule steady state path*.

20.E.9^A Argue that, for the one-consumer model of Section 20.D, the golden rule path cannot arise as part of a competitive equilibrium. [*Hint*: The key fact is that $\delta < 1$.]

20.F.1^C Consider two arbitrary functions $\gamma_1(w)$ and $\gamma_2(w)$ that are defined for $w > 0$, take nonnegative values, and satisfy $\gamma_1(w) + \gamma_2(w) = w$ for all w. Suppose also that they are twice continuously differentiable.

Show that for any $\alpha > 0$ there is a utility function for two commodities, $u(x_1, x_2)$, that is increasing and concave on the domain $\{(x_1, x_2): x_1 + x_2 \leq \alpha\}$ and is such that $(\gamma_1(w), \gamma_2(w))$ coincides with the Engel curve functions for prices $p_1 = 1$, $p_2 = 1$ and wealth $w < \alpha$. [*Hint*: Let $u(x_1, x_2) = (x_1 + x_2)^{1/2} - \varepsilon[(x_1 - \gamma_1(x_1 + x_2))^2 + (x_2 - \gamma_2(x_1 + x_2))^2]$ and take ε to be small enough. Verify then that $\nabla u(x_1, x_2)$ is strictly positive and $D^2u(x_1, x_2)$ is negative definite for any (x_1, x_2) such that $0 < x_1 + x_2 \leq \alpha$, and that the Engel curve is as required.]

20.F.2^A Suppose that, for $k \in \mathbb{R}_+$, the policy function $\psi(k)$ is a contraction (see the definition in the part of Section 20.F headed by "Low discount of time"). Draw several possible graphs for such a policy function and argue that there is always a unique steady state. Also, carry out the graphical dynamic analysis for your graphs and establish in this way that the steady states are always globally stable.

20.F.3^A Verify that for the classical Ramsey–Solow technology and for the cost-of-adjustment technology the cross derivative of uniform positive sign condition is satisfied.

20.F.4^A Carry out the verification concerning transitory shocks requested in Example 20.F.1.

20.F.5^A Carry out the verification concerning permanent shocks requested in Example 20.F.1.

20.G.1^B Analyze the equilibrium problem for the exchange case with two consumers (i.e., $I = 2$), and a single physical commodity (i.e., $L = 1$). Both consumers have the same discount factor [utility functions are of the form (20.B.1)]. In addition, assume that $\omega_{t1} + \omega_{t2} = 1$ for all t. Show in particular that the equilibrium consumption streams must be stationary, that the sequence of equilibrium prices is proportional (what is the rate of interest?), and that therefore there is only one stream of equilibrium consumptions.

20.G.2^A Consider an exchange model with two consumers. Utility functions are of the form (20.B.1) and both consumers have the same discount factor. There are no restrictions on the number of commodities L or on the total endowments at any t. Show that at a Pareto optimal allocation the following holds: for every consumer, the in-period marginal utilities of wealth of the consumer is the same across periods (and equal to the overall marginal utility of wealth of the consumer). Interpret and discuss what this means in terms of intertemporal and interindividual transfers of wealth.

20.G.3[B] The situation is the same as that of Exercise 20.G.2.

(a) Parametrize the Pareto frontier of the utility possibility set by the ratio of marginal utilities of wealth of the two consumers.

(b) Then express the equations of equilibrium à la Negishi (see Appendix A of Chapter 17). That is, write down one equation in one unknown (the ratio of marginal utilities of wealth) whose zeros are precisely the equilibria of the model.

(c) Argue in terms of the methodology discussed in Section 17.D that generically there are only a finite number of equilibria. Be as precise as you can.

20.G.4[A] Prove the claim made in footnote 32. Be explicit about the form of the equilibrium price sequences.

20.G.5[B] Verify that the concavity of the utility function implies that the expression (20.G.6) is larger than one in absolute value if there is no externality (i.e., if $\nabla_{23}^2 u(\cdot) = \nabla_{13}^2 u(\cdot) = 0$).

20.H.1[B] Show that in the context of Sections 20.D or 20.G (a finite number of consumers) no bubbles can arise at equilibrium.

20.H.2[B] In the framework of Section 20.H do the following (diagrammatic proofs are permissible).

(a) Show that if condition (20.H.3) is satisfied then, in the real asset case, the steady state is the only equilibrium.

(b) Show that if condition (20.H.3) is satisfied then, in the purely nominal asset case, the monetary steady state is the only equilibrium that is a Pareto optimum.

(c) Conversely, suppose that condition (20.H.3) is violated with strict inequality at $p_b = p_a$. Show then that, for the purely nominal asset case, there is more than one Pareto optimal equilibrium.

(d) (Harder) Suppose that the utility function is of the form $v(c_b) + \delta v(c_a)$. Investigate which conditions on $v(\cdot)$ and δ imply that the excess demand function satisfies condition (20.H.3). [*Hint*: Recall Example 17.F.2 for a special case.]

20.I.1[A] Verify the computation requested in the part of Section 20.I headed "The (short-run) law of demand in overlapping generations models."